Anisotropic Best τ_C-Approximation in Normed Spaces

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Abstract

The notions of the τ_C-Kolmogorov condition, the τ_C-sun and the τ_C-regular point are introduced, and the relationships between them and the best τ_C-approximation are explored. As a consequence, characterizations of best τ_C-approximations from some kind of subsets (not necessarily convex) are obtained. As an application, a characterization result for a set C to be smooth is given in terms of the τ_C-approximation.

Keywords: Minkowski function; the best τ_C-approximation; the τ_C-Kolmogorov condition; the τ_C-sun; smoothness.

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1 Introduction

Let X be a real normed linear space and C be a closed, bounded, convex subset of X having the origin as an interior point. Recall that the Minkowski function p_C with respect to the set C is defined by

$$p_C(x) = \inf\{t > 0 : x \in tC\}, \ \forall\, x \in X. \tag{1.1}$$

Let G be a subset of X and $x \in X$. Following [2], define the minimal time function $\tau_C(\cdot; G)$ by

$$\tau_C(x; G) := \inf_{g\in G} p_C(g - x), \ \forall x \in X. \tag{1.2}$$

The study of the minimal time function $\tau_C(\cdot; G)$ is motivated by its worldwide applications in many areas of variational analysis, optimization, control theory, approximation theory, etc., and has received a lot of attention; see e.g., [2,6,7,9,11–13,17,18]. In particular, the proximal subgradient of the minimal time function $\tau_C(\cdot; G)$ is estimated and computed in [7, 17, 18], for Hilbert spaces with applications to control theory; while other various subgradients such as the Fréchet subgradients, Clark subgradients, the ϵ-subdifferential as well as the limiting subdifferential of the minimal time function $\tau_C(\cdot; G)$ in Hilbert spaces and/or general Banach spaces are explored in [6, 9, 13].

Our interest in the present paper is focused on the following minimization problem, denoted by $\min(x, G)$,

$$\min_{g\in G} \ p_C(g - x), \tag{1.3}$$

where $x \in X$. Clearly, $g_0 \in G$ is a solution of the problem $\min(x, G)$ if and only if

$$p_C(g_0 - x) = \tau_C(x; G).$$

According to [2], any solution of the problem $\min(x, G)$ is called a best τ_C-approximation (or, generalized best approximation) to x from G. We denote by $\mathrm{P}_G^C(x)$ the set of all best τ_C-approximations to x from G. The generic well-posedness of the minimization problem $\min(x, G)$ in terms of the Baire category was studied in [2,11], while the relationships between the existence of solutions and directional derivatives of the function $\tau_C(x; G)$ was explored in [12].

In the special case when C is the closed unit ball $\mathbf{B}$ of X, the minimal time function (1.2) and the corresponding minimization problem (1.3) are reduced to the distance function of C and to the classical best approximation, respectively, which has been studied extensively and deeply, see, e.g., [3, 16, 21].

One aim of the present paper is to characterize the class of subsets of X for which the so-called τ_C-Kolmogorov condition holds about best τ_C-approximations. Another aim of the present paper is to prove the equivalence between the smoothness of the underlying set C and the convexity of τ_C-B-suns. In particular, by taking C to be the closed unit ball of X, our results extend the corresponding ones for nonlinear approximation problems; see, e.g., [3, 5, 21].

2 Preliminaries

Let X be a real normed linear space and let X^* denote its topological dual. Let A be a nonempty subset of X. As usual, we use bd$A$ and int$A$ to denote respectively the boundary and the interior of A. The polar of A is denoted by A° and defined by

$$A^\circ = \{x^* \in X^* : x^*(x) \le 1, \ \forall\ x \in A\}.$$

Then A° is a weakly*-closed convex subset of X^*. Furthermore, in the case when A is a convex bounded set with $0 \in \text{int } A$, A° is weakly*-compact with $0 \in \text{int } A^\circ$. In particular, $\mathbf{B}^\circ$ equals the closed unit ball of X^*. Moreover, for a set $A \subseteq X^*$, ext$A$ and $\overline{A}^*$ stand for the set of all extreme points and the weak*-closure of A, respectively. The following proposition is exactly the well-known Krein-Milman Theorem, see, e.g., [10].

Proposition 2.1. *Suppose that A is a compact convex subset of X^*. Then A equals the closed convex closure of* ext$A$.

Let

$$\mu = \inf_{\|x\|=1} p_C(x) \quad \text{and} \quad \nu = \sup_{\|x\|=1} p_C(x).$$

We end this section with some known and useful properties of the Minkowski function; see [15, Section 1] for assertions **(i)**-**(v)** while **(vi)** is an immediate consequence of **(iii)** and **(v)**.

Proposition 2.2. *Let $x, y \in X$ and $x^* \in X^*$. Then we have the following assertions.*

(i) $p_C(x) \ge 0$, *and* $p_C(x) = 0 \Longleftrightarrow x = 0$.

(ii) $p_C(x + y) \le p_C(x) + p_C(y)$.

(iii) $p_C(\lambda x) = \lambda p_C(x)$ *for each* $\lambda > 0$.

(iv) $p_C(x) \le 1 \Longleftrightarrow x \in C$.

(v) $p_C(x) = \sup_{x^* \in C^\circ} x^*(x)$ *and* $p_{C^\circ}(x^*) = \sup_{x \in C} x^*(x)$.

(vi) $\mu\|x\| \le p_C(x) \le \nu\|x\|$.

3 Characterization of the best τ_C-approximation

The notion of suns introduced by Efimov and Stechkin (cf. [8]) has proved to be rather important in nonlinear approximation theory; see, e.g., [3–5, 8, 21] and references therein. In the following definition we extend this notion to the case of the generalized approximation. Throughout the whole paper, we always assume that $G \subseteq X$ is a nonempty subset of X, and let $x \in X$ and $g_0 \in G$, unless specially stated.

Definition 3.1. *The element g_0 is called*

(a) *a τ_C-solar point of G with respect to x if $g_0 \in \mathrm{P}_G^C(x)$ implies that $g_0 \in \mathrm{P}_G^C(x_\lambda)$ for each $\lambda > 0$, where $x_\lambda = g_0 + \lambda(x - g_0)$;*

(b) *a τ_C-solar point of G if g_0 is a τ_C-solar point of G with respect to each $x \in X$.*

We say that G is a τ_C-sun of X if each point of G is a τ_C-solar point of G.

Remark 3.1. *We always write*

$$x_\lambda = g_0 + \lambda(x - g_0), \quad \forall \lambda > 0 \tag{3.1}$$

if no confusion caused. Thus

$$p_C(g_0 - x_\lambda) = \lambda p_C(g_0 - x), \quad \forall \lambda > 0. \tag{3.2}$$

Furthermore, the following implication holds (cf. [13, Lemma 3.1]:

$$g_0 \in \mathrm{P}_G^C(x) \Longrightarrow g_0 \in \mathrm{P}_G^C(x_\lambda), \ \forall \lambda \in [0, 1]. \tag{3.3}$$

For two points $x, y \in X$, we use $[x, y]$ to denote the closed interval with ends x and y, that is,

$$[x, y] := \{tx + (1 - t)y : \ t \in [0, 1]\}.$$

Recall (cf. [1]) that $g_0 \in G$ is a star-shaped point of G if $[g_0, g] \subseteq G$ for each $g \in G$. If G has a star-shaped point g_0, then G is called a star-shaped set with vertex g_0. Clearly, a convex subset is a star-shaped set with each vertex $g_0 \in G$. We use $\mathrm{S}(g_0, G)$ to denote the star-shaped set with vertex g_0 generated by G, that is,

$$\mathrm{S}(g_0, G) := \bigcup_{g \in G} [g_0, g].$$

Proposition 3.1. *Suppose that $g_0 \in G$ is a star-shaped point of G. Then g_0 is a τ_C-solar point of G. Consequently, any convex subset of X is a τ_C-sun of X.*

Proof. Let $x \in X$ be such that $g_0 \in \mathrm{P}_G^C(x)$. By Remark 3.1, we only need to show that $g_0 \in \mathrm{P}_G^C(x_\lambda)$ for each $\lambda > 1$. To this end, let $\lambda > 1$ and $g \in G$. Since g_0 is a star-shaped point of G, it follows from Proposition 2.2**(iii)** that

$$p_C(g_0 - x) \leq p_C\left(\left(\left(1 - \frac{1}{\lambda}\right) g_0 + \frac{1}{\lambda} g\right) - x\right) = \frac{1}{\lambda} p_C(g - x_\lambda).$$

By (3.2),

$$p_C(g_0 - x_\lambda) = \lambda p_C(g_0 - x) \leq p_C(g - x_\lambda).$$

Hence, $g_0 \in \mathrm{P}_G^C(x_\lambda)$. This shows that g_0 is a τ_C-solar point of G. □

The following proposition gives an equivalent condition for τ_C-solar points in terms of star-shaped points.

Proposition 3.2. *The element g_0 is a τ_C-solar point of G if and only if*

$$g_0 \in \mathrm{P}_G^C(x) \Longleftrightarrow g_0 \in \mathrm{P}_{\mathrm{S}(g_0,G)}^C(x), \quad \forall\, x \in X.$$

Proof. For $x \in X$, it is clear that $g_0 \in \mathrm{P}_{\mathrm{S}(g_0,G)}^C(x) \Longrightarrow g_0 \in \mathrm{P}_G^C(x)$. Thus, to complete the proof, it suffices to verify that g_0 is a τ_C-solar point of G if and only if

$$g_0 \in \mathrm{P}_G^C(x) \Longrightarrow g_0 \in \mathrm{P}_{\mathrm{S}(g_0,G)}^C(x) \tag{3.4}$$

holds for each $x \in X$. To this end, let $x \in X$. By Definition 3.1 and Remark 3.1, one has that g_0 is a τ_C-solar point of G if and only if the following implication holds:

$$g_0 \in \mathrm{P}_G^C(x) \Longrightarrow g_0 \in \mathrm{P}_G^C(x_{\frac{1}{1-\lambda}}), \quad \forall\, \lambda \in [0,1). \tag{3.5}$$

Note by Proposition 2.2**(iii)** that

$$\begin{aligned} & g_0 \in \mathrm{P}_G^C(x_{\frac{1}{1-\lambda}}),\ \forall\, \lambda \in [0,1) \\ \Longleftrightarrow\quad & p_C(g_0 - x) \le p_C((\lambda g_0 + (1-\lambda)g) - x),\ \forall g \in G,\ \forall\, \lambda \in [0,1) \\ \Longleftrightarrow\quad & g_0 \in \mathrm{P}_{S(g_0,G)}^C(x). \end{aligned}$$

Hence, (3.5) holds if and only if (3.4) holds. This completes the proof. □

Definition 3.2. *The element g_0 is called a local best τ_C-approximation to x from G if there exists an open neighborhood $U(g_0)$ of g_0 such that $g_0 \in \mathrm{P}_{G\cap U(g_0)}^C(x)$.*

Clearly, if $g_0 \in \mathrm{P}_G^C(x)$, then g_0 is a local best τ_C-approximation to x from G. The following proposition shows that the converse remains true if g_0 is a τ_C-solar point of G.

Proposition 3.3. *Suppose that g_0 is a τ_C-solar point of G. Then $g_0 \in \mathrm{P}_G^C(x)$ if and only if g_0 is a local best τ_C-approximation to x from G.*

Proof. The necessity part is clear as noted earlier. Below we prove the sufficiency part. To this end, suppose that g_0 is a local best τ_C-approximation to x from G. Then there exists an open neighborhood $U(g_0)$ of g_0 such that $g_0 \in \mathrm{P}_{U(g_0)\cap G}^C(x)$. We claim that there is $\lambda > 0$ such that $g_0 \in \mathrm{P}_G^C(x_\lambda)$. Indeed, otherwise, one has that $g_0 \notin \mathrm{P}_G^C(x_{1/n})$ for each $n \in \mathbb{N}$. Thus there exists a sequence $\{g_n\} \subseteq G$ such that, for each $n \in \mathbb{N}$,

$$p_C(g_n - x_{1/n}) < p_C(g_0 - x_{1/n}) = \frac{1}{n} p_C(g_0 - x). \tag{3.6}$$

It follows from Proposition 2.2**(vi)** that $\lim_{n\to\infty} g_n = g_0$. This implies that there exists $n_0 \in \mathbb{N}$ such that $g_{n_0} \in U(g_0) \cap G$. This together with (3.6) implies that $g_0 \notin \mathrm{P}_{U(g_0)\cap G}^C(x_{1/n_0})$, which is a contradiction by Remark 3.1 as $g_0 \in \mathrm{P}_{U(g_0)\cap G}^C(x)$. Hence the claim stands; that is $g_0 \in \mathrm{P}_G^C(x_\lambda)$ for some $\lambda > 0$. Noting that $x = g_0 + \frac{1}{\lambda}(x_\lambda - g_0)$ and that g_0 is a τ_C-solar point of G, we have that

$$g_0 \in \mathrm{P}_G^C\left(g_0 + \frac{1}{\lambda}(x_\lambda - g_0)\right) = \mathrm{P}_G^C(x)$$

and completes the proof. □

For the sequel study, we need to introduce the following notation:

$$\Sigma_{g_0-x} := \{x^* \in C^\circ : x^*(g_0 - x) = p_C(g_0 - x)\}.$$

Then Σ_{g_0-x} is a nonempty, weakly*-compact convex subset of C°. Furthermore, write $\mathcal{E}_{g_0-x} := \mathrm{ext}\Sigma_{g_0-x}$. Then, $\mathcal{E}_{g_0-x} \neq \emptyset$ by Proposition 2.1. Moreover, by definition, we have that

$$\mathcal{E}_{g_0-x} = \mathrm{ext}C^\circ \cap \Sigma_{g_0-x}. \tag{3.7}$$

The notions stated in Definition 3.3 below are extension of the notions of Kolmogorov Condition and Papini Condition in approximation theory to the setting of the best τ_C-approximation theory, see, e.g., [4, 5, 21] and [14].

Definition 3.3. *The pair (x, g_0) is said to satisfy*

(a) *the τ_C-Kolmogorov condition (the τ_C-KC, for short) if*

$$\max_{x^*\in\Sigma_{g_0-x}} x^*(g-g_0) \geq 0, \quad \forall\, g \in G; \tag{3.8}$$

(b) *the τ_C-Papini condition (the τ_C-PC, for short) if*

$$\max_{x^*\in\Sigma_{g-x}} x^*(g_0-g) \leq 0, \quad \forall\, g \in G. \tag{3.9}$$

Clearly, by (3.7), Σ_{g_0-x} in (3.8) and Σ_{g-x} in (3.9) can be replaced by $\mathcal{E}_{g_0-x}$ and $\mathcal{E}_{g-x}$, respectively.

In the case when C is the closed unit ball of X, that is, p_C is the norm, the notions of the regular point and the strongly regular point were introduced and studied respectively in [5] and [21]. The following notions of the τ_C-regular point and the strongly τ_C-regular point are respectively generalizations of the corresponding regular point and strongly regular point in best approximation theory.

Definition 3.4. *The element g_0 is called*

(a) *a τ_C-regular point of G with respect to x if for any weakly*-closed subset A of C° satisfying for some $g \in G$ the condition*

$$\mathcal{E}_{g_0-x} \subseteq A \subseteq \overline{\mathrm{ext} C^\circ}^* \quad \text{and} \quad \min_{x^*\in A} x^*(g_0-g) > 0, \tag{3.10}$$

there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \to g_0$ and

$$x^*(g_0-g_n) > x^*(g_0-x) - p_C(g_0-x), \quad \forall\, x^* \in A,\ \forall\, n \in \mathbb{N}; \tag{3.11}$$

(b) *a strongly τ_C-regular point of G with respect to x if for any weakly*-closed subset A of C° satisfying* (3.10) *for some $g \in G$, there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \to g_0$ and*

$$x^*(g_0-g_n) > 0, \quad \forall\, x^* \in A,\ \forall\, n \in \mathbb{N}. \tag{3.12}$$

We say that G is a τ_C-regular set (resp. strongly τ_C-regular set) of X if each point of G is a τ_C-regular point (resp. strongly τ_C-regular point) of G with respect to each $x \in X$.

Roughly speaking, a τ_C-regular point g_0 of G with respect to x means that, for any weakly*-closed neighbourhood A of the extreme set $\mathcal{E}_{g_0-x}$, if there exists some $g \in G$ such that $g_0 - g$ is positive on A, then any neighbourdhood of g_0 contains an element $g' \in G$ such that $g_0 - g'$ has the same sign as $g_0 - g$ on the set $\mathcal{E}_{g_0-x}$ while out of this set it can have opposite sign but have to be controlled within $p_C(g_0-x) - x^*(g_0-x)$; while, for a strongly τ_C-regular point g_0, $g_0 - g'$ must have the same sign as $g_0 - g$ on the whole weakly*-closed neighbourhood A.

Remark 3.2. *Clearly, the strong τ_C-regularity implies the τ_C-regularity. We don't know wether the converse is true even in the case when C is the closed unit ball of X.*

Remark 3.3. *Clearly, any interior point of G is a strongly τ_C-regular point of G. Moreover, it is not difficult to check that any star-shaped point of G is a strongly τ_C-regular point of G.*

Below we provide an example to illustrate the notions.

Example 3.1. *Let $X := \mathbb{R}^2$ be the 2-dimensional Euclidean space. Let C be the equilateral triangle with the vertexes $(0,2), (\sqrt{3},-1)$ and $(-\sqrt{3},-1)$ (see Figure 1). Then $0 \in \mathrm{int}C$ and C° is the equilateral triangle with vertexes x_1^*, x_2^*, x_3^*, where $x_1^* := \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $x_2^* := (0,-1)$ and $x_3^* := \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, and so $\mathrm{ext}C^\circ = \{x_1^*, x_2^*, x_3^*\}$ (see Figure 2). Let G be the subset defined by*

$$G = \left\{(t_1,t_2): \ t_2 \geq \frac{1}{2}t_1, \ t_1 \leq 0\right\} \bigcup \left\{(t_1,t_2): t_2 \geq -\frac{1}{2}t_1, \ t_1 \geq 0\right\}$$

(see Figure 3). Clearly, G is not convex. We shall show that G is a strongly τ_C-regular set. To this end, let $g_0 \in G$. By Remark 3.3, we assume, without loss of generality, that $g_0 = (a_1,a_2) \in \mathrm{bd}G$ satisfies that

$$a_1 > 0 \quad and \quad a_2 = -\frac{1}{2}a_1 \tag{3.13}$$

(noting that the origin is a star-shaped point of G). Let A be a weakly*-closed subset of C° satisfying (3.10) for some $g = (b_1,b_2) \in G$ and some $x \in X$. In the case when $b_1 \geq 0$ and $b_2 \geq -\frac{1}{2}b_1$, choose $g_n = (1-\frac{1}{n})g_0 + \frac{1}{n}g \in G$ for each $n \in \mathbb{N}$. Then it is easy to see that $\{g_n\}$ is as desired. Below we consider the case when $b_1 \leq 0$ and $b_2 \geq \frac{1}{2}b_1$. It follows from (3.13) that

$$(a_2-b_2) - \frac{1}{2}(a_1-b_1) = -a_1 + \frac{1}{2}b_1 - b_2 \leq -a_1 < 0.$$

Consequently,

$$x_3^*(g_0-g) = -\frac{\sqrt{3}}{2}(a_1-b_1) + \frac{1}{2}(a_2-b_2) \leq -\frac{\sqrt{3}}{2}(a_1-b_1) + \frac{1}{4}(a_1-b_1) = \frac{3}{8}(1-2\sqrt{3})a_1 < 0.$$

This means that $x_3^* \notin A$. Therefore, it suffices to consider the case when $A = \{x_1^*, x_2^*\}$. To this end, let $g_n = \left(1-\frac{1}{n}\right)g_0$ for each $n \in \mathbb{N}$. Then $\{g_n\} \subseteq G$ and $g_n \to g_0$. Noting that $x_1^*(g_0) = \left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right)a_1$ and $x_2^*(g_0) = \frac{1}{2}a_1$, we have that

$$\min_{x^* \in A} x^*(g_0-g_n) = \frac{1}{n}\min_{x^* \in A} x^*(g_0) = \frac{1}{2n}a_1 > 0, \quad \forall n \in \mathbb{N}.$$

This means that g_0 is a strongly regular point of G, and so G is a strongly τ_C-regular set.

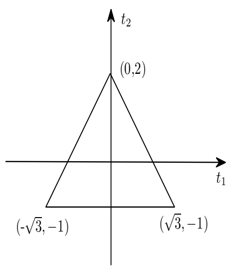

Figure 1

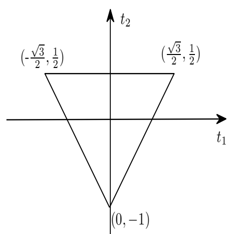

Figure 2

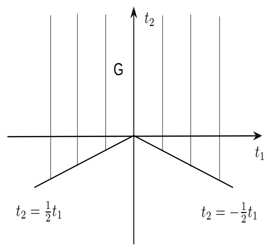

Figure 3

Theorem 3.1. *Consider the following assertion:*

(i) *The pair (x, g_0) satisfies the τ_C-KC.*

(ii) *The point $g_0 \in \mathrm{P}_G^C(x)$.*

(iii) *The pair (x, g_0) satisfies the τ_C-PC.*

Then **(i)**$\Longrightarrow$**(ii)**$\Longrightarrow$**(iii)***. In addition, if g_0 is a strongly τ_C-regular point of G with respect to x, then* **(i)**$\Longleftrightarrow$**(ii)**$\Longleftrightarrow$**(iii)***.*

Proof. **(i)**$\Longrightarrow$**(ii)**. Suppose that **(i)** holds and let $g \in G$. Then by Definition 3.3 **(a)** , there exists $x^* \in \Sigma_{g_0-x}$ such that $x^*(g - g_0) \geq 0$. This together with Proposition 2.2 **(v)** implies that

$$p_C(g_0 - x) = x^*(g_0 - g) + x^*(g - x) \leq x^*(g - x) \leq p_C(g - x);$$

hence, $g_0 \in \mathrm{P}_G^C(x)$ as $g \in G$ is arbitrary and **(ii)** is proved.

(ii)$\Longrightarrow$**(iii)**. Suppose that **(ii)** holds. Let $g \in G$ and $x^* \in \Sigma_{g-x}$. Then $x^*(g - x) = p_C(g - x)$; hence

$$x^*(g_0 - g) = x^*(g_0 - x) + x^*(x - g) \leq p_C(g_0 - x) - p_C(g - x) \leq 0.$$

This shows that (x, g_0) satisfies the τ_C-PC and **(iii)** is proved.

Suppose that g_0 is a strongly τ_C-regular point of G with respect to x. It suffices to prove the implication **(iii)**$\Longrightarrow$**(i)**. To this end, suppose on the contrary that (x, g_0) does not satisfy the τ_C-KC. Then there exist $g \in G$ and $\delta > 0$ such that

$$\max_{x^* \in \Sigma_{g_0-x}} x^*(g - g_0) = -\delta. \tag{3.14}$$

Let

$$U = \{x^* \in \overline{\mathrm{ext} C^\circ}^* : x^*(g - g_0) < -\frac{\delta}{2}\} \quad \text{and} \quad A = \overline{U}^*. \tag{3.15}$$

Then A is a weakly*-closed subset of C° satisfying $\mathcal{E}_{g_0-x} \subseteq A \subseteq \overline{\mathrm{ext} C^\circ}^*$ and

$$\min_{x^* \in A} x^*(g_0 - g) \geq \frac{\delta}{2} > 0. \tag{3.16}$$

Since g_0 is a strongly τ_C-regular point of G with respect to x, there exists a sequence $\{g_n\} \subseteq G$ such that

$$\lim_{n \to \infty} g_n = g_0 \tag{3.17}$$

and

$$\min_{x^* \in A} x^*(g_0 - g_n) > 0, \quad \forall\, n \in \mathbb{N}. \tag{3.18}$$

Let $K = \overline{\mathrm{ext} C^\circ}^* \setminus U$. Then K is the weakly*-compact subset of $\overline{\mathrm{ext} C^\circ}^*$. Moreover, we have that $K \cap \Sigma_{g_0-x} = \emptyset$ because $\mathcal{E}_{g_0-x} \subset U$ by (3.14) and (3.15). Consequently,

$$\alpha := \max_{x^* \in K} x^*(g_0 - x) < p_C(g_0 - x). \tag{3.19}$$

Set

$$\alpha_0 := \frac{1}{2}(p_C(g_0 - x) - \alpha). \tag{3.20}$$

As $\lim_{n\to\infty} g_n = g_0$ by (3.17), one has from Proposition 2.2 **(vi)** that there exists $n_0 \in \mathbb{N}$ such that

$$\max\{p_C(g_{n_0} - g_0), p_C(g_0 - g_{n_0})\} < \alpha_0. \tag{3.21}$$

Thus

$$p_C(g_0 - x) \le p_C(g_0 - g_{n_0}) + p_C(g_{n_0} - x) < \alpha_0 + p_C(g_{n_0} - x). \tag{3.22}$$

It follows from Proposition 2.2 **(v)**, (3.19)-(3.22) that

$$\begin{aligned} \max_{x^*\in K} x^*(g_{n_0} - x) &\le \max_{x^*\in K} x^*(g_{n_0} - g_0) + \max_{x^*\in K} x^*(g_0 - x) \\ &\le p_C(g_{n_0} - g_0) + \alpha \\ &< \alpha_0 + \alpha \\ &= p_C(g_0 - x) - \alpha_0 \\ &< p_C(g_{n_0} - x). \end{aligned}$$

This shows that $K \cap \mathcal{E}_{g_{n_0}-x} = \emptyset$, and so $\mathcal{E}_{g_{n_0}-x} \subseteq \overline{\text{ext}C^\circ}^* \setminus K = U$. Therefore,

$$\max_{x^*\in\mathcal{E}_{g_{n_0}-x}} x^*(g_0 - g_{n_0}) \ge \inf_{x^*\in U} x^*(g_0 - g_{n_0}) = \min_{x^*\in A} x^*(g_0 - g_{n_0}) > 0$$

thanks to (3.18). Thus, $\max_{x^*\in\Sigma_{g_{n_0}-x}} x^*(g_0 - g_{n_0}) > 0$ by (3.7). This contradicts the τ_C-PC for the pair (x, g_0), and the proof is complete. □

Remark 3.4. *One natural and interesting question is: Wether the implication* **(iii)⟹(i)** *remains true under simple (not strong) regularity assumption? Even in the case when C is the unit closed ball, we don't know the answer and so we leave it open.*

The main result of this section is a generalization of [4, Theorem 9].

Theorem 3.2. *The following assertions are equivalent.*

(i) *The element g_0 is a τ_C-solar point of G with respect to x.*

(ii) *The element $g_0 \in \mathrm{P}_G^C(x)$ if and only if (x, g_0) satisfies the τ_C-KC.*

(iii) *The element g_0 is a τ_C-regular point of G with respect to x.*

Proof. **(i)⟹(ii)**. Suppose that **(i)** holds. The sufficiency part of **(ii)** follows directly from Theorem 3.1. Below we show the necessity part of **(ii)**. To this end, we assume that $g_0 \in \mathrm{P}_G^C(x)$. Without loss of generality, we may assume that $x \ne g_0$ and $p_C(g_0 - x) \ne 0$. Let $g \in G \setminus \{g_0\}$ be arbitrary. Then $g_0 \in \mathrm{P}_{[g_0,g]}^C(x)$ by Proposition 3.2 and $[g_0, g] \cap \mathrm{int}(C + x) = \emptyset$. Thus, by the separation theorem (cf. [10]), there exist $y^* \in X^* \setminus \{0\}$ and a real number r such that

$$y^*(z - x) \ge r, \quad \forall\, z \in [g_0, g] \tag{3.23}$$

and

$$y^*(y - x) \le r, \quad \forall\, y \in C + x. \tag{3.24}$$

In particular, we have that $p_{C^\circ}(y^*) \le r$ by Proposition 2.2**(v)**. Let $x^* = r^{-1}y^*$. Then $p_{C^\circ}(x^*) \le 1$ and so $x^* \in C^\circ$. Since $g_0 \in [g_0, g] \cap (C + x)$, it follows from (3.23) and (3.24) that

$$r = y^*(g_0 - x). \tag{3.25}$$

This implies that $x^*(g_0-x)=1=p_C(g_0-x)$, and so $x^*\in\Sigma_{g_0-x}$. Furthermore, $x^*(g-g_0)\geq 0$ thanks to (3.23) and (3.25). Hence the necessity part holds as $g\in G\setminus\{g_0\}$ is arbitrary and the implication is proved.

(ii)$\Longrightarrow$(i). Suppose that **(ii)** holds and assume that $g_0\in \mathrm{P}_G^C(x)$. Then

$$\max_{x^*\in\Sigma_{g_0-x}} x^*(g-g_0)\geq 0,\quad \forall\, g\in G. \tag{3.26}$$

Let $\lambda>0$ be arbitrary. Noting that $g_0-x_\lambda=\lambda(g_0-x)$, one has that $\Sigma_{g_0-x_\lambda}=\Sigma_{g_0-x}$. This together with (3.26) implies that (x_λ,g_0) satisfies the τ_C-KC, and so $g_0\in\mathrm{P}_G^C(x_\lambda)$. This shows that g_0 is a τ_C-solar point of G with respect to x.

(ii)$\Longrightarrow$(iii). Suppose that **(ii)** holds. Let $g\in G$ and A be a weakly*-closed subset of C° satisfying (3.10). Then

$$\max_{x^*\in\mathcal{E}_{g_0-x}} x^*(g-g_0)\leq \max_{x^*\in A} x^*(g-g_0)<0.$$

This together with (3.7) implies that (x,g_0) does not satisfy the τ_C-KC; hence, $g_0\notin\mathrm{P}_G^C(x)$ by **(ii)**. Since g_0 is a τ_C-solar point of G with respect to x by the equivalence of **(i)** and **(ii)** just proved, it follows from Proposition 3.3 that g_0 is not local best τ_C-approximation to x from G. Thus there exists a sequence $\{g_n\}\subseteq G$ such that $g_n\to g_0$ and

$$p_C(g_n-x)<p_C(g_0-x),\quad \forall\, n\in\mathbb{N}. \tag{3.27}$$

Let $x^*\in A$ be arbitrary. Then, $x^*(g_n-x)\leq p_C(g_n-x)$ for each $n\in\mathbb{N}$. This and (3.27) imply that

$$x^*(g_0-g_n)\geq x^*(g_0-x)-p_C(g_n-x)>x^*(g_0-x)-p_C(g_0-x),\quad \forall\, n\in\mathbb{N}.$$

In view of Definition 3.4, g_0 is a τ_C-regular point of G.

(iii)$\Longrightarrow$(ii). Suppose that **(iii)** holds. By Theorem 3.1 **(i)**, it suffices to prove that (x,g_0) satisfies the τ_C-KC whenever $g_0\in\mathrm{P}_G^C(x)$. To this end, we assume on the contrary that it is not the case. Then, there are $g\in G$ and $\delta>0$ such that (3.14) holds. Let U and A be defined by (3.15). Then (3.16) holds. Since g_0 is a τ_C-regular point of G with respect to x, there exists a sequence $\{g_n\}\subseteq G$ such that $\lim_{n\to\infty} g_n=g_0$ and (3.11) holds. This together with Proposition 2.2 implies that

$$\max_{x^*\in A} x^*(g_n-x)<p_C(g_0-x),\quad \forall\, n\in\mathbb{N}. \tag{3.28}$$

On the other hand, let $K=\overline{\mathrm{ext}C^\circ}^*\setminus U$. Then there is $\epsilon>0$ such that

$$\max_{x^*\in K} x^*(g_0-x)<p_C(g_0-x)-\epsilon. \tag{3.29}$$

Since $\lim_{n\to\infty} g_n=g_0$, one has that $\lim_{n\to\infty} p_C(g_n-g_0)=0$. Let $n_0\in\mathbb{N}$ be such that $p_C(g_{n_0}-g_0)<\epsilon$. It follows from (3.29) that

$$\max_{x^*\in K} x^*(g_{n_0}-x)\leq p_C(g_{n_0}-g_0)+\max_{x^*\in K} p_C(g_0-x)<p_C(g_0-x). \tag{3.30}$$

Combining (3.28) and (3.30), we obtain that $p_C(g_{n_0}-x)<p_C(g_0-x)$, which contradicts that $g_0\in\mathrm{P}_G^C(x)$. The proof is complete. □

The following corollary is a global version of Theorem 3.2.

Corollary 3.1. *The following statements are equivalent.*

(i) *G is a τ_C-sun of X.*

(ii) *For each $g_0 \in G$ and each $x \in X$, $g_0 \in \mathrm{P}_G^C(x)$ if and only if (x, g_0) satisfies the τ_C-KC.*

(iii) *G is a τ_C-regular set.*

4 Smoothness and convexity of τ_C-B-suns

We begin with the notion of smooth convex sets (cf. [10]). Let $x \in \mathrm{bd}C$ and $x^* \in C^\circ$. Recall that x^* is a supporting functional of C at x if $x^*(x) = 1$.

Definition 4.1. *The set C is called smooth if each point of $\mathrm{bd}C$ has a unique supporting functional.*

The following notion extends a similar concept introduced in [3].

Definition 4.2. *The set G is called a τ_C-B-sun of X if for each $x \in X$ there exists $g_0 \in \mathrm{P}_G^C(x)$ such that g_0 is a τ_C-solar point of G with respect to x.*

Remark 4.1. *Clearly, if G is a existence set (i.e., $\mathrm{P}_G^C(x) \neq \emptyset$ for each $x \in X$), then a τ_C-sun must be τ_C-B-sun. The converse is not true in general, see [21, Example 1.4] for the case when C is the unit ball of X.*

The main result of this section is as follows, which extends [3, Theorem 2.5] to the setting of the best τ_C-approximation.

Theorem 4.1. *The set C is smooth if and only if each τ_C-B-sun of X is convex.*

Proof. "$\Longrightarrow$". Suppose that C is smooth and G is a τ_C-B-sun of X. It suffices to verify that $\frac{1}{2}(g_1 + g_2) \in G$ for each pair of elements $g_1, g_2 \in G$. To this end, let $g_1, g_2 \in G$ and let $x = \frac{1}{2}(g_1 + g_2)$. By Definition 4.2 there exists $g_0 \in \mathrm{P}_G^C(x)$ such that g_0 is a τ_C-solar point of G with respect to x. It follows from Theorem 3.2 that there exist $x_1^*, x_2^* \in \Sigma_{g_0 - x}$ such that

$$x_i^*(g_i - g_0) \geq 0, \quad \forall\, i = 1, 2. \tag{4.1}$$

Suppose that $g_0 \neq x$ and consider the point $\bar{y} := (g_0 - x)/p_C(g_0 - x)$. Then $\bar{y} \in C$ and $x_i^*(\bar{y}) = 1$ for each $i = 1, 2$. Noting that each $x_i^* \in C^\circ$, we have that x_i^*, $i = 1, 2$, are supporting functionals of C° at $\bar{y}$. Hence, $x_1^* = x_2^*$ by the smoothness of C. Let $x^* = x_1^*$. Then

$$p_C(g_0 - x) = x^*(g_0 - x) = \frac{1}{2}x^*(g_0 - g_1) + \frac{1}{2}x^*(g_0 - g_2) \leq 0$$

thanks to (4.1). By Proposition 2.2 **(i)**, one has $x = g_0$, which is a contradiction, and hence $\frac{1}{2}(g_1 + g_2) \in G$.

"$\Longleftarrow$". Conversely, suppose on the contrary that C is not smooth. Then there exist $x_0 \in \mathrm{bd}C$ and two functional $x_1^*, x_2^* \in C^\circ$ such that

$$x_1 \neq x_2^* \quad \text{and} \quad x_1^*(x_0) = x_2^*(x_0) = 1. \tag{4.2}$$

Let $G_i := \{x \in X : x_i^*(x) \geq 0\}$ for each $i = 1, 2$, and let $G = G_1 \cup G_2$. Then G is a closed and nonconvex subset of X. In fact, the closedness is clear. To prove the nonconvexity, we first prove that $\ker(x_1^*) \setminus G_2 \neq \emptyset$, where $\ker(x^*) := \{x \in X : x^*(x) = 0\}$ is the kernel of the functional x^*. Indeed, otherwise, one has that $\ker(x_1^*) \subseteq G_2$. Let $x \in X$. Then, by (4.2),

$$x - x_i^*(x)x_0 \in \ker(x_i^*), \quad \forall\, i = 1, 2. \tag{4.3}$$

In particular, we have that $x - x_1^*(x)x_0 \in G_2$. It follows that

$$x_2^*(x) \geq x_2^*(x_0)x_1^*(x) = x_1^*(x), \ \forall\, x \in X.$$

This implies that $x_1^* = x_2^*$, which is a contradiction. Therefore, $\ker(x_1^*) \setminus G_2 \neq \emptyset$. Similarly, we also have that $\ker(x_2^*) \setminus G_1 \neq \emptyset$. Take $x_1 \in \ker(x_1^*) \setminus G_2$ and $x_2 \in \ker(x_2^*) \setminus G_1$. Then $x_1, x_2 \in G$ and $x_i^*(\frac{1}{2}x_1 + \frac{1}{2}x_2) < 0$ for each $i = 1, 2$. This means that $\frac{1}{2}x_1 + \frac{1}{2}x_2 \notin G$, and so G is not convex.

By the definition of τ_C, it is easy to see that

$$\tau_C(x; G) = \min\{\tau_C(x; G_1), \tau_C(x; G_2)\}, \quad \forall\, x \in X. \tag{4.4}$$

We will prove that, for each $i = 1, 2$,

$$\tau_C(x; G_i) = -x_i^*(x), \quad \forall\, x \in X \setminus G_i. \tag{4.5}$$

To this end, fix $i = 1, 2$ and let $x \in X \setminus G_i$. Then $x_i^*(x) < 0$, and by (4.3), one has that

$$\tau_C(x; G_i) \leq p_C((x - x_i^*(x)x_0) - x) = -x_i^*(x)p_C(x_0) = -x_i^*(x)$$

(noting that $p_C(x_0) = 1$). On the other hand,

$$\tau_C(x; G_i) = \inf_{g \in G_i} p_C(g - x) \geq \inf_{g \in G_i} x_i^*(g - x) \geq -x_i^*(x),$$

and the assertion (4.5) is seen to hold.

Below we prove that G is a τ_C-B-sun of X. To this end, let $x \in X \setminus G$ and $d := \tau_C(x; G)$. Without loss of generality, we may assume that

$$\tau_C(x; G_1) \leq \tau_C(x; G_2). \tag{4.6}$$

Then by (4.5),

$$d = \tau_C(x; G_1) = -x_1^*(x). \tag{4.7}$$

Let $g_0 = x + dx_0$. Then by (4.3)

$$g_0 = x - x_1^*(x)x_0 \in \ker(x_1^*) \subseteq G_1 \subseteq G. \tag{4.8}$$

Moreover

$$p_C(g_0 - x) = dp_C(x_0) = d = \tau_C(x; G_1) = \tau_C(x; G). \tag{4.9}$$

Hence $g_0 \in \mathrm{P}_G^C(x)$. We assert that $g_0 \in \mathrm{P}_G^C(x_\lambda)$ for each $\lambda > 0$. Granting this, G is a nonconvex, τ_C-B-sun and the proof of Theorem 4.1 is complete. Let $\lambda > 0$. By (4.8) and (4.9), we have that $g_0 \in \mathrm{P}_{G_1}^C(x)$. Since G_1 is convex, it follows from Proposition 3.1 that

$g_0 \in \mathrm{P}^C_{G_1}(x_\lambda)$. Thus, to complete the proof, it suffices to show that $\tau_C(x_\lambda; G) = \tau_C(x_\lambda; G_1)$. To do this, note that $x_2^*(x) \le x_1^*(x) = -d$ by (4.5), (4.6) and (4.7). Consequently,

$$\begin{aligned} x_2^*(x_\lambda) &= (1-\lambda)x_2^*(g_0) + \lambda x_2^*(x) \\ &= (1-\lambda)(x_2^*(x) + dx_2^*(x_0)) + \lambda x_2^*(x) \\ &= x_2^*(x) + d(1-\lambda) \\ &\le -d + d(1-\lambda) \\ &= -\lambda d. \end{aligned}$$

This together with (4.5) (with x_λ in place of x) implies that

$$\tau_C(x_\lambda; G_2) \ge d\lambda = \tau_C(x_\lambda; G_1).$$

Therefore, $\tau_C(x_\lambda; G) = \tau_C(x_\lambda; G_1)$ thanks to (4.4). The proof is complete. □

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