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Semi-Infinite Optimization under Convex Function Perturbations: Lipschitz Stability

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Abstract This paper is devoted to the study of the stability of the solution map for the parametric convex semi-infinite optimization problem under convex function perturbations in short, PCSI. We establish sufficient conditions for the pseudo-Lipschitz property of the solution map of PCSI under perturbations of both objective function and constraint set. The main result obtained is new even when the problems under consideration reduce to linear semi-infinite optimization. Examples are given to illustrate the obtained results.

Keywords Convex programming · Semi-infinite optimization · Solution map · Lipschitz stability · Slater constraint qualification

1 Introduction

We will denote by $\mathbf{CO}(\mathbb{R}^n)$ the set of all the finite convex functions on $\mathbb{R}^n$. Let T be nonempty compact subset of a metric space. $\mathbf{C}_1(\mathbb{R}^n \times T)$ be the set of all continuous function $g\colon \mathbb{R}^n \times T \to \mathbb{R}$, such that $g_t(\cdot) := g(\cdot, t)$ is convex for all $t \in T$.

Consider *parametric convex semi-infinite optimization* problem PCSI under functional perturbations of both objective function and constraint set on the parameter

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space

$$P := \mathbf{CO}(\mathbb{R}^n) \times \mathbf{C}_1(\mathbb{R}^n \times T)$$

formulated as follows: For each the parameter $p := (f, g)$, we have the *convex semi-infinite optimization* problem

$$(\mathrm{CSI})_p\colon \quad \min f(x) \quad \text{subject to } x \in \mathcal{C}(p),$$

where

$$\mathcal{C}(p) = \{x \in \Omega \mid g_t(x) := g(x,t) \le 0,\ \forall t \in T\}$$

is the set of *feasible points* of $(\mathrm{CSI})_p$ and Ω is a closed convex subset of $\mathbb{R}^n$.

We aim at studying the recent development of parametric convex semi-infinite optimization. Namely, we will investigate the pseudo-Lipschitz property of the solution map of PCSI. It is well known that semi-infinite optimization problems naturally arise in approximation theory, optimal control, and in numerous engineering problems. Many papers are published every year on theory, methods and applications of semi-infinite programming and its extensions; see, e.g., [1–22] and the references therein for more comments and discussions. Especially, solution stability, such as lower semicontinuity, upper semicontinuity, pseudo-Lipschitz property, metric regularity, has attracted much attention of researchers in last years (see [1–10, 12–17, 21, 22]).

In parametric linear semi-infinite optimization, sufficient conditions for the lower and upper semi-continuity of the minimal value functions under perturbations of both objective function and constraint set have been given by B. Brosowski in [2]. M.J. Cánovas et al. [3] derived other characterizations of the same properties of the solution map and optimal value function together with the Lipschitz property of optimal value function. Furthermore, the pseudo-Lipschitz property of the solution map was investigated in [8] for parametric linear vector semi-infinite optimization problems under linear perturbations of the objective function and continuous perturbations of the right-hand side of the constraints. In the framework of parameter convex semi-infinite optimization problems, sufficient conditions for the metric regularity of the inverse of the solution map (or, equivalently, for the pseudo-Lipschitz property of solution map) under continuous perturbations of the right-hand side of the constraints and linear perturbations of the objective function was presented in [5].

The main goal of this paper is to establish sufficient conditions for the pseudo-Lipschitz property of the solution map of PCSI

$$\mathcal{S}(p) := \{x^* \in \mathcal{C}(p) \mid f(x^*) \le f(x),\ \forall x \in \mathcal{C}(p)\}$$

under convex functional perturbations of both objective function and constraint set. The main result obtained in the paper is new even when the problems under consideration reduce to linear semi-infinite optimization. Our result extends the corresponding result in [5] for convex semi-infinite optimization problems under canonical perturbations.

The paper is organized as follows. In Sect. 2, we recall some basic definitions and preliminaries from convex analysis and set-valued analysis, and give some auxiliary results, which will be useful in next section. In Sect. 3, we present sufficient conditions for the pseudo-Lipschitz property of the solution map of PCSI under convex functional perturbations of both objective function and constraint set. Moreover, examples are given to illustrate the obtained results. An application to parametric linear semi-infinite optimization problems is presented in Sect. 4.

2 Preliminaries and Auxiliary Results

Let us first recall some standard notions from convex analysis and set-valued analysis; see, e.g., [19, 23–25]. Let (X, d) be a metric space. Given $M \subset X$, the closure and the interior of M are denoted by $\operatorname{cl} M$ and $\operatorname{int} M$, respectively. We will use $\mathcal{N}(x)$ to denote the set of all neighborhoods of $x \in X$. The distance from $x \in X$ to M is defined by

$$d(x, M) := \inf\{\operatorname{dist}(x, y) \mid y \in M\},$$

where $\operatorname{dist}(x, y)$ denotes the distance between two points x and y, and $d(x, \varnothing) := +\infty$.

In particular, when $X = \mathbb{R}^n$ $(n \in \mathbb{N})$, we will denote by $\operatorname{co} M$ and $\operatorname{cone} M$ the convex hull and the convex conical hull of M, respectively. Write $\operatorname{co} \emptyset = \emptyset$ and $\operatorname{cone} \emptyset = \{0_n\}$, where 0_n is the zero vector of $\mathbb{R}^n$. $\mathbb{B}$ stands for the open unit ball in $\mathbb{R}^n$, and $\mathbb{B}(x, \rho) := x + \rho\mathbb{B}$.

Let $F : X \rightrightarrows Y$ be a multifunction between metric spaces. The effective domain and the graph of the multifunction F are given, respectively, by

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}, \qquad \operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Definition 2.1

(i) F is said to be *closed* at the point $x^0 \in X$ iff for all sequences $\{x^i\}$ in X and $\{y^i\}$ in Y satisfying $x^i \to x^0$, $y^i \to y^0$, $y^i \in F(x^i)$ for all $i \in \mathbb{N}$, one has $y^0 \in F(x^0)$.

(ii) F is *upper semicontinuous* (usc for brevity) at $x^0 \in X$ iff for every open set V containing $F(x^0)$ there exists $U_0 \in \mathcal{N}(x^0)$ such that $F(x) \subset V$ for all $x \in U_0$.

(iii) F is said to be *lower semicontinuous* (lsc for brevity) at $x^0 \in \operatorname{dom} F$ iff for any open set $V \subset Y$ satisfying $V \cap F(x^0) \neq \emptyset$ there exists $U_0 \in \mathcal{N}(x^0)$ such that $V \cap F(x) \neq \emptyset$ for all $x \in U_0$.

(iv) F is said to be *continuous* at $x^0 \in \operatorname{dom} F$ iff it is both upper semicontinuous and lower semicontinuous at x^0.

(v) F is *pseudo-Lipschitz* (also called *Aubin continuous* or *Lipschitz-like*) at $(x^0, y^0) \in \operatorname{gph} F$ iff there exist $U \in \mathcal{N}(x^0)$, $V \in \mathcal{N}(y^0)$ and a constant $\ell > 0$, such that

$$d(y^2, F(x^1)) \leq \ell d(x^1, x^2),$$

for all $x^1, x^2 \in U$, and all $y^2 \in V \cap F(x^2)$.

Now we define, as in [20], a distance between convex functions in $\mathbf{CO}(\mathbb{R}^n)$. Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact sets in $\mathbb{R}^n$, such that

$$K_m \subset \operatorname{int} K_{m+1} \quad \text{and} \quad \mathbb{R}^n = \bigcup_{m=1}^{\infty} K_m. \tag{1}$$

In particular, we could have considered $K_m = m(\operatorname{cl}\mathbb{B})$. Let $f, g \in \mathbf{CO}(\mathbb{R}^n)$. First, for each $m = 1, 2, \ldots$, we define

$$\delta_m(f, g) := \sup_{x \in K_m} \{|f(x) - g(x)|\}.$$

Then we define a metric σ on $\mathbf{CO}(\mathbb{R}^n)$ by

$$\sigma(f, g) := \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\delta_m(f, g)}{1 + \delta_m(f, g)} \quad \forall f, g \in \mathbf{CO}(\mathbb{R}^n), \tag{2}$$

which describes the topology of the uniform convergence of convex functions on compact sets.

Lemma 2.1 [20, Proposition 3.1] *Let* $f \in \mathbf{CO}(\mathbb{R}^n)$ *and* $\{f^k\}_{k=1}^{\infty} \subset \mathbf{CO}(\mathbb{R}^n)$. *Then,* $\sigma(f^k, f) \to 0$ *as* $k \to \infty$, *if and only if* f^k *converges uniformly to* f *on any compact subset of* $\mathbb{R}^n$.

Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. Denote by $\partial h(\bar{x})$ the *subdifferential* of f at $\bar{x} \in \operatorname{dom} f$,

$$\partial h(\bar{x}) := \{u^* \in \mathbb{R}^n \mid h(x) - h(\bar{x}) \geq \langle u^*, x - \bar{x}\rangle,\ \forall x \in \mathbb{R}^n\}.$$

We denote by $N_\Omega(\bar{x})$, the normal cone to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \operatorname{cl}\Omega$,

$$N_\Omega(\bar{x}) := \{u^* \in \mathbb{R}^n \mid \langle u^*, x - \bar{x}\rangle \leq 0,\ \forall x \in \Omega\}.$$

The *set of active constraints* at $x \in \mathcal{C}(p)$ is defined by

$$T_p(x) = \{t \in T \mid g_t(x) := g(x, t) = 0\}.$$

Lemma 2.2 *The multifunction* $\mathcal{T} : P \times \mathbb{R}^n \rightrightarrows T$, *such that* $\mathcal{T}(p, x) = T_p(x)$, *is usc at every* $(p^0, x^0) \in P \times \mathbb{R}^n$.

Proof The proof is straightforward and so is omitted. □

Let $p \in P$. We say, as in [12], that p satisfies the Slater constraint qualification iff there exists $\hat{x} \in \Omega$ such that

$$g(\hat{x}, t) < 0 \quad \forall t \in T.$$

Lemma 2.3 *Let $p=(f,g)\in P$. Then, p satisfies the Slater constraint qualification, if and only if $0_n\notin \operatorname{co}(\{\partial g_t(\bar{x})\mid t\in T_p(\bar{x})\})+N_\Omega(\bar{x})$ for all $\bar{x}\in\mathcal{C}(p)$ with $T_p(\bar{x})\neq\emptyset$.*

Proof Define the function $h:\mathbb{R}^n\to\mathbb{R}$ by $h(x):=\max_{t\in T} g_t(x)$. Obviously, h is convex on $\mathbb{R}^n$ and

$$\mathcal{C}(p)=\{x\in\Omega\mid h(x)\leq 0\}.$$

Since h is a finite-valued convex function, it is continuous. Moreover, for each $\bar{x}\in\mathcal{C}(p)$ with $T_p(\bar{x})\neq\emptyset$, we have

$$h(\bar{x}):=\max_{t\in T} g_t(\bar{x})=0.$$

Then p satisfies the Slater constraint qualification, if and only if there is no $\bar{x}\in\mathcal{C}(p)$ with $T_p(\bar{x})\neq\emptyset$ such that it is a minimizer of h on Ω. This is equivalent to

$$0_n\notin\partial h(\bar{x})+N_\Omega(\bar{x}). \tag{3}$$

It follows from [19, Theorem VI.4.4.2] that

$$\partial h(\bar{x})=\operatorname{co}(\{\partial g_t(\bar{x})\mid t\in T_p(\bar{x})\}). \tag{4}$$

Combining (3) and (4) we obtain $0_n\notin\operatorname{co}(\{\partial g_t(\bar{x})\mid t\in T_p(\bar{x})\})+N_\Omega(\bar{x})$. □

Next, we define the distance between functions in $\mathbf{C}_1(\mathbb{R}^n\times T)$, which is given by

$$\sigma_\infty(g,\bar{g}):=\sup_{t\in T}\sigma(g_t,\bar{g}_t)\quad \text{for all } g,\bar{g}\in\mathbf{C}_1(\mathbb{R}^n\times T), \tag{5}$$

where $g_t(x):=g(x,t)$, $\forall x\in\mathbb{R}^n$ $\forall t\in T$. Then $(\sigma_\infty,\mathbf{C}_1(\mathbb{R}^n\times T))$ is a complete metrizable space. It is well known that the convergence of a sequence $\{g^k\}_{k=1}^\infty\subset\mathbf{C}_1(\mathbb{R}^n\times T)$ to $g\in\mathbf{C}_1(\mathbb{R}^n\times T)$ describes the uniform convergence, on T, of the functions $g^k(x,\cdot)$ to $g(x,\cdot)$ for every $x\in\mathbb{R}^n$.

From now on, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. The distance between two elements $p=(f,g)$ and $\bar{p}=(\bar{f},\bar{g})$ belonging to the parameter space $P:=\mathbf{CO}(\mathbb{R}^n)\times\mathbf{C}_1(\mathbb{R}^n\times T)$ is formulated by

$$d_\infty(p,\bar{p}):=\max\left\{\sigma(f,\bar{f}),\sigma_\infty(g,\bar{g})\right\}, \tag{6}$$

where σ and σ_∞ are defined as in (2) and (5), respectively.

Lemma 2.4 *Let $p^0=(f^0,g^0)\in P$. If p^0 satisfies the Slater constraint qualification, then $\mathcal{C}\colon P\rightrightarrows\mathbb{R}^n$ is lower semicontinuous at p^0.*

Proof Let $p^0=(f^0,g^0)\in P$. Let V be an open convex set such that $V\cap\mathcal{C}(p^0)\neq\emptyset$. Since p^0 satisfies the Slater constraint qualification and T is compact, there must exist an element $\hat{x}\in\mathcal{C}(p^0)$ and $\rho>0$, such that

$$g^0(\hat{x},t)\leq-\rho\quad\forall t\in T. \tag{7}$$

Take $x \in V \cap \mathcal{C}(p^0)$ and choose a number $r \in]0, 1]$ such that

$$x_r := x + r(\hat{x} - x) \in V.$$

By the convexity of $\mathcal{C}(p^0)$ and V, $x_{r'} \in V \cap \mathcal{C}(p^0)$ for all $r' \in [0, r]$. It follows from (7) that

$$g^0(x_{r'}, t) \le (1 - r')g^0(x, t) + r'g^0(\hat{x}, t) \le -r'\rho \quad \forall t \in T,\ \forall r' \in [0, r]. \tag{8}$$

It remains to prove that there exists $r_1 > 0$ such that, for every $\bar{p} := (\bar{f}, \bar{g}) \in P$ satisfying $d_\infty(\bar{p}, p^0) < r_1\rho$,

$$V \cap C(\bar{p}) \ne \emptyset. \tag{9}$$

Take an arbitrary $\varepsilon > 0$. Let

$$\mathbb{B}(p^0, \rho\varepsilon) := \{p \in P \mid d_\infty(p, p^0) < \rho\varepsilon\}. \tag{10}$$

Take any $\bar{p} := (\bar{f}, \bar{g}) \in \mathbb{B}(p^0, \rho\varepsilon)$. Let $\{K_m\}$ be a sequence of compact subsets of $\mathbb{R}^n$ defined as in (1). Then there exists $j \in \mathbb{N}$ such that $x_r \in K_j$ for all $r \in [0, 1]$. Let $t \in T$. Since the function $\tau \to \frac{\tau}{1+\tau}$ is increasing on $[0, +\infty[$, it follows from (1) that

$$0 \le \frac{1}{2^j}\frac{\delta_j(\bar{g}_t, g_t^0)}{1 + \delta_j(\bar{g}_t, g_t^0)} \le \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{\delta_i(\bar{g}_t, g_t^0)}{1 + \delta_i(\bar{g}_t, g_t^0)} = \sigma(\bar{g}_t, g_t^0) \le \sigma_\infty(\bar{g}, g^0).$$

Obviously, $\sigma_\infty(\bar{g}, g^0) \le d_\infty(\bar{p}, p^0)$. Hence,

$$0 \le \frac{1}{2^j(1 + \delta_j(\bar{g}_t, g_t^0))}\frac{\delta_j(\bar{g}_t, g_t^0)}{\rho} \le \frac{d_\infty(\bar{p}, p^0)}{\rho} < \varepsilon.$$

This implies $\frac{\delta_j(\bar{g}_t, g_t^0)}{\rho} \to 0$ as $\varepsilon \to 0$ for all $t \in T$. Therefore, there must exist $r_1 > 0$ and $r_0 \in]0, r[$ such that $\delta_j(\bar{g}_t, g_t^0) \le r_0\rho$ for all $t \in T$ whenever $d_\infty(\bar{p}, p^0) \le r_1\rho$. Hence,

$$\delta_j(\bar{g}_t, g_t^0) \le r_0\rho < r\rho \quad \forall t \in T. \tag{11}$$

Let $x_s := x + s(\hat{x} - x)$, $s \in [r_0, r]$. Clearly, $x_s \in V \cap \mathcal{C}(p^0)$. We have $x_s \in \mathcal{C}(\bar{p})$. Indeed, it follows from (11) that, for every $t \in T$,

$$\bar{g}(x_s, t) - g^0(x_s, t) \le \sup_{x \in K_j} |\bar{g}(x, t) - g^0(x, t)| =: \delta_j(\bar{g}_t, g_t^0) \le r_0\rho.$$

Then, by (8),

$$\bar{g}(x_s, t) \le g^0(x_s, t) + r_0\rho \le -(s - r_0)\rho \le 0 \quad \forall t \in T.$$

This implies $x_s \in \mathcal{C}(\bar{p})$ whenever $d_\infty(\bar{p}, p^0) \le r_1\rho$. Thus $\mathcal{C}$ is lower semicontinuous at p. The proof is complete. □

Lemma 2.5 *Let $p^0 = (f^0, g^0) \in P$. Suppose that the following conditions hold*:

(i) p^0 *satisfies the Slater constraint qualification*;
(ii) $\mathcal{S}(p^0) = \{x^0\}$.

Then $\mathcal{S}$ is lower semicontinuous at p^0.

Proof We first show that $\mathcal{C}$ is closed at p^0. Let $\{p^k = (f^k, g^k)\}_{k=1}^{\infty} \subset P$ and $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ be sequences such that $x^k \in \mathcal{C}(p^k)$, $x^k \to x^0$ and $p^k \to p^0$ as $k \to \infty$. It is sufficient to show that $x^0 \in \mathcal{C}(p^0)$. From Lemma 2.1 and the compactness of T, it follows that, for any $\varepsilon > 0$, there exist a $\rho > 0$ and a positive integer k_0 such that

$$|g^k(x,t) - g^0(x^0,t)| < \frac{\varepsilon}{2} \quad \forall x \in \mathbb{B}(x^0,\rho),\ \forall k \geq k_0,\ \forall t \in T.$$

This implies that

$$g^0(x^0,t) < \frac{\varepsilon}{2} + g^k(x^k,t) \leq \frac{\varepsilon}{2} \quad \forall k \geq k_0,\ \forall t \in T.$$

Letting $\varepsilon \to 0$, we have

$$g^0(x^0,t) \leq 0 \quad \forall t \in T.$$

Since $x^k \in \mathcal{C}(p^k) \subset \Omega$ for all k implies $x^0 \in \Omega$, we have $x^0 \in \mathcal{C}(p^0)$.

We next prove that there is $U(p^0) \in \mathcal{N}(p^0)$ such that

$$\mathcal{S}(p) \neq \emptyset \quad \text{for all } p \in U(p^0). \tag{12}$$

Indeed, if our claim were false then there would exist $\{p^k = (f^k, g^k)\}_{k=1}^{\infty} \in P$ converging to p^0, such that

$$\mathcal{S}(p^k) = \emptyset \quad \text{for all } k \geq 1. \tag{13}$$

By the lower semicontinuity of C at p^0, there would exist $\lambda > 0$, $y^k \in \mathcal{C}(p^k) \cap \operatorname{cl}\mathbb{B}(x^0,\lambda)$ such that $y^k \to x^0$ as $k \to \infty$. For each $k = 1, 2, \ldots$, since $\mathcal{S}(p^k) = \emptyset$ it follows that there must exist $z^k \in \mathcal{C}(p^k) \setminus \mathbb{B}(x^0,\lambda)$ such that

$$f^k(z^k) - f^k(y^k) < 0 \tag{14}$$

(otherwise for all $z \in \mathcal{C}(p^k) \setminus \mathbb{B}(x^0,\lambda)$, $f^k(z) \geq f^k(y^k)$. This implies that f^k has a minimizer on $\mathcal{C}(p^k) \cap \operatorname{cl}\mathbb{B}(x^0,\lambda)$). Consider the following two possible cases:

(a) A subsequence of $\{z^k\}$ converges. We can assume that $\lim_{k\to\infty} z^k = z^0$. Letting $k \to \infty$ in (14), we obtain from the closeness of $\mathcal{C}$ at p^0 that $z^0 \in \mathcal{C}(p^0)$ and

$$f^0(z^0) - f^0(x^0) \leq 0.$$

Hence, $z^0 = x^0$, which contradicts $z^0 \in \mathcal{C}(p^0) \setminus \mathbb{B}(x^0,\lambda)$.

(b) $\lim_{k\to\infty} \|z^k\| = +\infty$. Without loss of generality, we can assume that

$$\lim_{k\to\infty} \frac{z^k}{\|z^k\|} = \hat{z}, \qquad \|\hat{z}\| = 1.$$

By the convexity of f^k, we have

$$f^k\left(\frac{1}{\|z^k\|}z^k + \left(1 - \frac{1}{\|z^k\|}\right)y^k\right) - f^k(y^k) \leq \frac{1}{\|z^k\|}(f^k(z^k) - f^k(y^k)).$$

It follows from (14) that

$$f^k\left(\frac{1}{\|z^k\|}z^k + \left(1 - \frac{1}{\|z^k\|}\right)y^k\right) - f^k(y^k) < 0. \tag{15}$$

Obviously, $\frac{1}{\|z^k\|}z^k + (1 - \frac{1}{\|z^k\|})y^k \in \mathcal{C}(p^k)$ for all $k \geq 1$. Letting $k \to \infty$ in (15), we have

$$\hat{z} + x^0 \in \mathcal{C}(p^0) \quad \text{and} \quad f^0(\hat{z} + x^0) - f^0(x^0) \leq 0.$$

Hence $\hat{z} + x^0 = x^0$, which is impossible. Combining cases (a) and (b), we have proved (12).

It remains to prove that $\mathcal{S}$ is lower semicontinuous at p^0. Let $\{p^k = (f^k, g^k)\}_{k=1}^{\infty} \subset P$ be a sequence such that $p^k \to p^0$ as $k \to \infty$. Then, by (12), there exist $k_0 \geq 1$ and $x^k \in \mathcal{C}(p^k)$ such that

$$x^k \in \mathcal{S}(p^k) \quad \text{for all } k \geq k_0.$$

Since $\mathcal{C}$ is lower semicontinuous at p^0, it follows that for each $k = 1, 2, \ldots$ there exists $w^k \in \mathcal{C}(p^k)$ such that $\lim_{k\to\infty} w^k = x^0$. Clearly,

$$f^k(x^k) - f^k(w^k) \leq 0 \quad \text{for all } k \geq k_0. \tag{16}$$

Then $\{x^k\}_{k=1}^{\infty}$ is bounded. Indeed, if $\lim_{k\to\infty} \|x^k\| = +\infty$ then, we can assume that

$$\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = \hat{x}, \qquad \|\hat{x}\| = 1.$$

It follows from the convexity of f^k that

$$f^k\left(\frac{1}{\|x^k\|}x^k + \left(1 - \frac{1}{\|x^k\|}\right)w^k\right) - f^k(w^k) \leq \frac{1}{\|x^k\|}(f^k(x^k) - f^k(w^k)).$$

Letting $k \to \infty$, we can assert from (16) that

$$\hat{x} + x^0 \in \mathcal{C}(p^0) \quad \text{and} \quad f^0(\hat{x} + x^0) - f^0(x^0) \leq 0.$$

This implies that $\hat{x} + x^0 = x^0$, a contradiction. Hence, $\{x^k\}_{k=1}^{\infty}$ is bounded.

Without loss of generality, we can assume that $\lim_{k\to\infty} x^k = \bar{x} \in \mathcal{C}(p^0)$. It follows from (16) that $f^0(\bar{x}) - f^0(x^0) \leq 0$. Thus, $\bar{x} = x^0$. The proof is complete. □

Final in this section, we recall a important result from [24] on the convergence of subdifferentials of convex functions.

Lemma 2.6 [24, Theorem 24.5] *Let $\varphi \in \mathbf{CO}[\mathbb{R}^n]$ and $\{\varphi^k\}_{k=1}^{\infty} \subset \mathbf{CO}[\mathbb{R}^n]$ such that φ^k converges pointwise to φ on an open convex set $D \subset \mathbb{R}^n$ as $k \to \infty$. Let $x \in D$ and $\{x^k\}_{k=1}^{\infty} \subset D$ converging to x. Then, for each $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that*

$$\partial\varphi^k(x^k) \subset \partial\varphi(x) + \varepsilon B \quad \text{for all } k \geq k_0.$$

3 Lipschitz Stability of the Solution Map

In this section we present sufficient conditions for the pseudo-Lipschitz property of the solution set mapping $\mathcal{S}$ of PCSI at the nominal parameter.

We first recall a optimality condition for PCSI at a given point.

Lemma 3.1 *Let $p^0 = (f^0, g^0) \in P$. Suppose that p^0 satisfies the Slater constraint qualification. Then x^0 is a solution of PCSI, if and only if*

$$-\partial f(x^0) \cap \left[\operatorname{cone}\left(\bigcup_{t\in T(x^0)} \partial g_t(x^0)\right) + N_\Omega(x^0)\right] \neq \emptyset.$$

Proof It is immediate from Theorem 4.1 in [22]. □

From the above optimality condition we can establish the following result, which is useful in the sequel. $|T|$ denotes the cardinality of T.

Proposition 3.1 *Let $p^0 = (f^0, g^0) \in P$ and $x^0 \in \mathcal{S}(p^0)$. Suppose that the following conditions hold*:

(i) *p^0 satisfies the Slater constraint qualification*;
(ii) *There is no $T_0 \subset T_{p^0}(x^0)$ with $|T_0| < n$ satisfying*

$$-\partial f^0(x^0) \cap \left[\operatorname{cone}\left(\bigcup_{t\in T_0} \partial g_t^0(x^0)\right) + N_\Omega(x^0)\right] \neq \emptyset.$$

Then, the following statements are valid:

(a) *for any $\{(p^k, x^k) = (f^k, g^k, x^k)\}_{k=1}^{\infty} \subset \operatorname{gph}\mathcal{S}$ which converges to $(p^0, x^0) = (f^0, g^0, x^0)$ in $\operatorname{gph}\mathcal{S}$, there exist $u^k \in \partial f^k(x^k)$, $t_i^k \in T_{p^k}(x^k)$, $u_i^k \in \partial g_{t_i^k}^k(x^k)$ and $\lambda_i^k > 0$ for $i \in \{1, 2, \ldots, n\}$, such that*

$$-u^k - \sum_{i=1}^{n} \lambda_i^k u_i^k \in N_\Omega(x^k) \quad \text{for } k \text{ large enough}$$

and $\{u_1^k, \ldots, u_n^k\}$ forms a basis of $\mathbb{R}^n$;

(b) $\mathcal{S}(p^0) = \{x^0\}$.

Proof (a) It is easily seen that p satisfies the Slater constraint qualification in a some neighborhood of p^0. Let $\{(p^k, x^k) = (f_k, g^k, x^k)\}_{k=1}^{\infty}$ be a sequence of gph $\mathcal{S}$ such that $\{(p^k, x^k)\}$ converges to $(p^0, x^0) = (f^0, g^0, x^0) \in \operatorname{gph} \mathcal{S}$. Since $(p^k, x^k) \to (p^0, x^0)$ as $k \to \infty$, it follows that p^k satisfies the Slater constraint qualification for k large enough. Applying Lemma 3.1, we can assert from the Carathéodory's theorem that, for k large enough, there exist $q \in \mathbb{N}$, $u^k \in \partial f^k(x^k)$, $t_i^k \in T_{p^k}(x^k)$, $u_i^k \in \partial g^k_{t_i^k}(x^k)$ and $\lambda_i^k > 0$ for $i \in \{1, 2, \dots, q\}$, such that $q \leq n$,

$$-u^k - \sum_{i=1}^{q} \lambda_i^k u_i^k \in N_\Omega(x^k), \tag{17}$$

and $\{u_i^k \mid i = 1, \dots, q\}$ is a linearly independent system.

It remains to prove that $q = n$. Conversely, suppose that $q < n$. By the compactness of T, we can assume, by taking a subsequence if necessary, that $\{t_i^k\}$ converges to some $t_i \in T$ for each $i \in \{1, \dots, q\}$. Since $t_i^k \in T_{p^k}(t_i^k)$, it follows from Lemma 2.2 that $t_i \in T_{p^0}(x^0)$.

We claim that, for each $i \in \{1, \dots, q\}$, there exists $\lambda_i \geq 0$ such that

$$\lim_{k\to\infty} \lambda_i^k = \lambda_i. \tag{18}$$

Indeed, if our claim were false then, by taking a subsequence if necessary, we can assume that there exists $i_0 \in \{1, \dots, q\}$ such that

$$\lim_{k\to\infty} \lambda_{i_0}^k = +\infty.$$

Put $\mu^k := \sum_{i=1}^{q} \lambda_i^k$, $k \geq 1$. Then $\lim_{k\to\infty} \mu^k = +\infty$, and there is no loss of generality in assuming that the sequence $\{\frac{\lambda_i^k}{\mu^k}\}_{k \geq k_0}$ converges to some $\mu_i \geq 0$ for each $i \in \{1, \dots, q\}$. Dividing by μ^k in (17) and letting $k \to \infty$, we deduce from assumption (i), the robustness of the normal cone [10, p. 58], Lemma 2.6 and the compactness of the subgradient of a convex function that

$$-\sum_{i=1}^{q} \mu_i u_i \in N_\Omega(x^0) \quad \text{with} \ \sum_{i=1}^{q} \mu_i = 1$$

and $u_i \in \partial g^0_{t_i}(x^0)$. This means that $0_n \in \operatorname{co}(\{\partial g_t^0(x^0) \mid t \in T_{p^0}(x^0)\}) + N_\Omega(x^0)$, which contradicts the assertion of Lemma 2.3, and (18) follows.

Letting $k \to \infty$ in (17), we get $u \in \partial f^0(x^0)$, $u_i \in \partial g^0_{t_i}(x^0)$ $(i = 1, 2, \dots, q)$,

$$-u - \sum_{i=1}^{q} \mu_i u_i \in N_\Omega(x^0) \quad \text{with } \{t_1, \dots, t_q\} \subset T_{p^0}(x^0) \text{ and } q < n, \tag{19}$$

which contradicts assumption (ii). Thus, $q = n$.

(b) Since $x^0 \in \mathcal{S}(p^0)$, it follows from the Carathéodory's theorem and assumption (ii) that there exist $u \in \partial f^0(x^0)$, $t_i \in T_{p^0}(x^0)$, $u_i \in \partial g^0_{t_i}(x^0)$ and $\lambda_i > 0$ for $i \in \{1, \dots, n\}$, such that $\{u_1, \dots, u_n\}$ is a basis of $\mathbb{R}^n$ and

$$-u - \sum_{i=1}^{n} \lambda_i u_i \in N_\Omega(x^0). \tag{20}$$

Take any

$$y \in \mathcal{S}(p^0).$$

Then, for each $i \in \{1, \dots, n\}$, $g^0_{t_i}(x^0) = 0$ and

$$\langle u_i, y - x^0 \rangle \le g^0_{t_i}(y) - g^0_{t_i}(x^0) = g^0_{t_i}(y).$$

Hence,

$$\langle u_i, y - x^0 \rangle \le 0. \tag{21}$$

Besides,

$$\left\langle -u - \sum_{i=1}^{n} \lambda_i u_i, y - x^0 \right\rangle \le 0.$$

Therefore,

$$0 = f^0(y) - f^0(x^0) \ge \langle u, y - x^0 \rangle \ge -\sum_{i=1}^{n} \lambda_i \langle u_i, y - x^0 \rangle \ge 0.$$

This implies that $\sum_{i=1}^{n} \lambda_i \langle u_i, y - x^0 \rangle = 0$ and so, $\langle u_i, y - x^0 \rangle = 0$ for every $i \in \{1, \dots, n\}$. Since $\{u_1, \dots, u_n\}$ is a basis of $\mathbb{R}^n$, it follows that there exist the real numbers β_i, $i \in \{1, \dots, n\}$, such that $y - x^0 = \sum_{i=1}^{n} \beta_i u_i$. Hence,

$$(\|y - x^0\|_n)^2 = \langle y - x^0, y - x^0 \rangle = \sum_{i=1}^{n} \beta_i \langle u_i, y - x^0 \rangle = 0.$$

Thus $y = x^0$. The proof is complete. □

Note that in [6] the combination of both conditions (i) and (ii) of Proposition 3.1 is referred to as *extended Nürnberger condition*; for more details and discussions we refer the reader to [5–7].

Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact sets of $\mathbb{R}^n$ defined as in (1). The following lemma is useful in the sequel.

Lemma 3.2 *Let Θ be a compact subset of $\mathbb{R}^n$ and let $\{x^k\}_{k=1}^{\infty}$ be a sequence belonging to $\Theta \setminus \{0\}$. Let $\{f^k\}_{k=1}^{\infty} \subset$ **CO**$(\mathbb{R}^n)$ and $f \in$ **CO**$(\mathbb{R}^n)$. If $\lim_{k\to\infty} \frac{\sigma(f^k, f)}{\|x^k\|} = 0$ then, for each $m \in \mathbb{N}$ such that $\Theta \subset K_m$, one has $\lim_{k\to\infty} \frac{\delta_m(f^k, f)}{\|x^k\|} = 0$.*

Proof Let $m \in \mathbb{N}$ be sufficient large such that $\Theta \subset K_m$. Since the function $\gamma(r) := \frac{r}{1+r}$ is increasing on $\mathbb{R}_+ := [0, +\infty[$, we have

$$\frac{\delta_m(f^k, f)}{1+\delta_m(f^k, f)} \le \frac{\delta_{m+k}(f^k, f)}{1+\delta_{m+k}(f^k, f)} \quad \text{for every } k \ge 1.$$

Obviously,

$$\sum_{i=m+1}^{\infty} \frac{1}{2^i} \frac{\delta_m(f^k, f)}{1+\delta_m(f^k, f)} \le \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\delta_j(f^k, f)}{1+\delta_j(f^k, f)} =: \sigma(f^k, f).$$

Hence

$$0 \le \frac{1}{2^m} \frac{\delta_m(f^k, f)}{1+\delta_m(f^k, f)} \le \sigma(f^k, f)$$

and so,

$$0 \le \frac{1}{2^m(1+\delta_m(f^k, f))} \frac{\delta_m(f^k, f)}{\|x^k\|} \le \frac{\sigma(f^k, f)}{\|x^k\|}.$$

Combining this with Lemma 2.1 we obtain the aimed conclusion. □

We now state the main result.

Theorem 3.1 *Let $p^0 = (f^0, g^0) \in P$ and $x^0 \in \mathcal{S}(p^0)$. Suppose that the following conditions hold*:

(i) *p^0 satisfies the Slater constraint qualification*;
(ii) *There is no $T_0 \subset T_{p^0}(x^0)$ with $|T_0| < n$ satisfying*

$$-\partial f^0(x^0) \cap \left[\text{cone}\left(\bigcup_{t\in T_0} \partial g_t^0(x^0)\right) + N_\Omega(x^0)\right] \ne \emptyset.$$

Then $\mathcal{S}$ is pseudo-Lipschitz at (p^0, x^0).

Proof Let $p^0 = (f^0, g^0) \in P$ and $x^0 \in \mathcal{S}(p^0)$. Suppose, in the contrary to our claim, that there exist a sequence $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ converging to x^0, the sequences $\{p^k = (f^k, g^k)\}_{k=1}^{\infty}$ and $\{\bar{p}^k = (\bar{f}^k, \bar{g}^k)\}_{k=1}^{\infty}$ in P, both converging to p^0, such that $x^k \in \mathcal{S}(p^k)$ and

$$d(x^k, \mathcal{S}(\bar{p}^k)) > k d_\infty(p^k, \bar{p}^k) \quad \text{for all } k \ge 1. \tag{22}$$

Since $\mathcal{S}$ is lower semicontinuous at p^0 by Lemma 2.5, it follows that there exists $\bar{x}^k \in \mathcal{S}(\bar{p}^k)$ satisfying $\bar{x}^k \to x^0$ as $k \to +\infty$. Hence, by (22), for each $k \ge 1$ and $x^k \ne \bar{x}^k$, we have

$$\max\left\{\frac{\sigma(f^k, \bar{f}^k)}{\|x^k - \bar{x}^k\|}, \frac{\sigma_\infty(g^k, \bar{g}^k)}{\|x^k - \bar{x}^k\|}\right\} = \frac{d_\infty(p^k, \bar{p}^k)}{\|x^k - \bar{x}^k\|} < \frac{1}{k}. \tag{23}$$

From Proposition 3.1, it follows that, for k large enough, there exist $u^k \in \partial f^k(x^k)$, $\bar{u}^k \in \partial \bar{f}^k(\bar{x}^k)$, $t_i^k \in T_{p^k}(x^k)$, $\bar{t}_i^k \in T_{\bar{p}^k}(\bar{x}^k)$, $u_i^k \in \partial g^k_{t_i^k}(x^k)$, $\bar{u}_i^k \in \partial \bar{g}^k_{\bar{t}_i^k}(\bar{x}^k)$, $\lambda_i^k > 0$, $\bar{\lambda}_i^k > 0$ for all $i = 1, 2, \dots, n$, such that both $\{u_1^k, \dots, u_n^k\}$ and $\{\bar{u}_1^k, \dots, \bar{u}_n^k\}$ form bases of $\mathbb{R}^n$ and

$$-u^k - \sum_{i=1}^n \lambda_i^k u_i^k \in N_\Omega(x^k), \qquad -\bar{u}^k - \sum_{i=1}^n \bar{\lambda}_i^k \bar{u}_i^k \in N_\Omega(\bar{x}^k). \tag{24}$$

By taking a subsequence if necessary, we can assume that, for each $i \in \{1, \dots, n\}$, the sequences $\{t_i^k\}_{k \geq k_0}$ and $\{\bar{t}_i^k\}_{k \geq k_0}$ converge to some t_i and $\bar{t}_i$, respectively. Then, by Lemma 2.2, we obtain

$$t_i,\ \bar{t}_i \in T_{p^0}(x^0), \quad i \in \{1, 2, \dots, n\}.$$

As in the proof of Proposition 3.1, we can assume that $\{\lambda_i^k\}_{k \geq k_0}$ and $\{\bar{\lambda}_i^k\}_{k \geq k_0}$ converge to some λ_i and $\bar{\lambda}_i$, respectively. Letting $k \to \infty$ in (24), we deduce from assumption (i), the robustness of the normal cone [23, p. 58], Lemma 2.6, and the compactness of the subgradient of any finite-valued convex function that, by taking a subsequence if necessary, $\lim_{k \to \infty} u^k = u \in \partial f^0(x^0)$, $\lim_{k \to \infty} \bar{u}^k = \bar{u} \in \partial f^0(x^0)$, $\lim_{k \to \infty} \bar{u}_i^k = \bar{u}_i \in \partial g^0_{t_i}(x^0)$ with $i \in \{1, 2, \dots, n\}$, and

$$-u - \sum_{i=1}^n \lambda_i u_i \in N_\Omega(x^0), \qquad -\bar{u} - \sum_{i=1}^n \bar{\lambda}_i \bar{u}_i \in N_\Omega(x^0). \tag{25}$$

From the Carathéodory's theorem and assumption (ii), it follows that $\lambda_i > 0, \bar{\lambda}_i > 0$ for all $i = 1, \dots, n$, and both $\{u_1, \dots, u_n\}$ and $\{\bar{u}_1, \dots, \bar{u}_n\}$ must be basis of $\mathbb{R}^n$.

Let $\{K_m\}_{m=1}^\infty$ be a sequence of compact subset of $\mathbb{R}^n$ defined as in (1). Then there must exist $m \in \mathbb{N}$ such that

$$\{\bar{x}^k - x^k\}_{k=1}^\infty, \{x^k\}_{k=1}^\infty, \{\bar{x}^k\}_{k=1}^\infty \subset K_m.$$

On the hand, since $t_i^k \in T_{p^k}(x^k)$ and $\bar{x}^k \in \mathcal{S}(\bar{p}^k) \subset \mathcal{C}(\bar{p}^k)$ for every $i \in \{1, \dots, n\}$, it follows that

$$g^k_{t_i^k}(x^k) = 0,\ \bar{g}^k_{t_i^k}(\bar{x}^k) \leq 0.$$

Hence,

$$\langle u_i^k, \bar{x}^k - x^k \rangle \leq g_{t_i^k}(\bar{x}^k) - g_{t_i^k}(x^k) \leq g_{t_i^k}(\bar{x}^k) - \bar{g}_{t_i^k}(\bar{x}^k) \leq \delta_m(g^k_{t_i^k}, \bar{g}^k_{t_i^k}).$$

Therefore,

$$\left\langle u_i^k, \frac{\bar{x}^k - x^k}{\|x^k - \bar{x}^k\|} \right\rangle \leq \frac{\delta_m(g^k_{t_i^k}, \bar{g}^k_{t_i^k})}{\|x^k - \bar{x}^k\|}. \tag{26}$$

Furthermore,

$$\sigma(g_t^k, \bar{g}_t^k) \leq \sup_{t \in T} \sigma(g_t^k, \bar{g}_t^k) =: \sigma_\infty(g^k, \bar{g}^k) \leq d_\infty(p^k, \bar{p}^k) \quad \forall t \in T.$$

This and (23) imply

$$\lim_{k\to\infty}\frac{\sigma(g^k_{t^k_i},\bar{g}^k_{t^k_i})}{\|x^k-\bar{x}^k\|}=0\quad\forall i\in\{1,2,\dots,n\}. \tag{27}$$

By taking a subsequence if necessary, we can assume that $\{\frac{\bar{x}^k-x^k}{\|x^k-\bar{x}^k\|_n}\}_{k\geq k_0}$ converges to some $z\in\mathbb{R}^n$ with $\|z\|=1$. Letting $k\to\infty$ in (26), we obtain from (27) and Lemma 3.2 that

$$\langle u_i,z\rangle\leq 0\quad\forall i\in\{1,\dots,n\}. \tag{28}$$

Besides, the first inclusion in (24) implies

$$\left\langle -u^k-\sum_{i=1}^n\lambda_i^k u_i^k,\bar{x}^k-x^k\right\rangle\leq 0. \tag{29}$$

Dividing both sides of (29) by $\|x^k-\bar{x}^k\|$ and letting $k\to\infty$, we can assert from (28) that

$$\langle -u,z\rangle\leq\sum_{i=1}^n\lambda_i\langle u_i,z\rangle\leq 0. \tag{30}$$

On the another hand, since $\bar{t}^k_i\in T_{\bar{p}^k}(\bar{x}^k)$ and $x^k\in\mathcal{S}(p^k)\subset\mathcal{C}(p^k)$ for every $i\in\{1,\dots,n\}$, it follows that

$$\bar{g}^k_{\bar{t}^k_i}(\bar{x}^k)=0,\qquad g^k_{\bar{t}^k_i}(x^k)\leq 0.$$

This implies that

$$\left\langle\bar{u}^k_i,\frac{x^k-\bar{x}^k}{\|\bar{x}^k-x^k\|}\right\rangle\leq\frac{\delta_m(\bar{g}^k_{t^k_i},g^k_{t^k_i})}{\|\bar{x}^k-x^k\|}.$$

In the same manner we see that, for every $i\in\{1,\dots,n\}$,

$$\langle\bar{u}_i,-z\rangle\leq 0,\qquad\langle -\bar{u},-z\rangle\leq\sum_{i=1}^n\bar{\lambda}_i\langle\bar{u}_i,-z\rangle.$$

Hence,

$$\langle\bar{u}_i,z\rangle\geq 0,\qquad\langle -\bar{u},z\rangle\geq\sum_{i=1}^n\lambda_i\langle\bar{u}_i,z\rangle\geq 0. \tag{31}$$

By the convexity of f^k and $\bar{f}^k$, we have

$$\langle u_k,\bar{x}^k-x^k\rangle\leq f^k(\bar{x}^k)-f^k(x^k)$$

and

$$\langle\bar{u}_k,x^k-\bar{x}^k\rangle\leq\bar{f}^k(x^k)-\bar{f}^k(\bar{x}^k).$$

Therefore,

$$\begin{aligned}\langle u_k, \bar{x}^k - x^k\rangle + \langle \bar{u}_k, x^k - \bar{x}^k\rangle &\le f^k(\bar{x}^k) - \bar{f}^k(\bar{x}^k) + \bar{f}^k(x^k) - f^k(x^k)\\ &\le 2\delta_m(f_k, \bar{f}_k).\end{aligned} \tag{32}$$

Dividing both sides of (32) by $\|x^k - \bar{x}^k\|$ and letting $k \to \infty$, we can assert from (23) and Lemma 3.2 that

$$\langle u, z\rangle + \langle \bar{u}, -z\rangle \le 0,$$

and so,

$$\langle -\bar{u}, z\rangle \le \langle -u, z\rangle. \tag{33}$$

Combining (28), (30), (31), and (33) we conclude that, for every $i \in \{1, \dots, n\}$,

$$\langle \bar{u}_i, z\rangle = \langle u_i, z\rangle = 0.$$

Since both $\{u_1^k, \dots, u_n^k\}$ and $\{\bar{u}_1^k, \dots, \bar{u}_n^k\}$ are basis of $\mathbb{R}^n$, it follows that $z = 0$, a contradiction. The proof is complete. □

The next example illustrates Theorem 3.1.

Example 3.1 For PCSI, let $T = [1, 2]$, $P := \mathbf{CO}(\mathbb{R}) \times \mathbf{C}_1(\mathbb{R} \times T)$ and $\Omega := \mathbb{R}$. Let $p^0 = (f^0, g^0) \in P$ be defined by $f^0(x) = x^2 + 3x + 2 \forall x \in \mathbb{R}$, and

$$g_t^0(x) = -x - t \quad \forall t \in T,\ x \in \mathbb{R}.$$

Then $\mathcal{C}(p^0) = [-1, +\infty[$. Let $x^0 = -1 \in \mathcal{S}(p^0)$. We now check the assumptions of Theorem 3.1. Clearly, $\hat{x} = 3$ is a Slater element for p^0 and so, assumption (i) is fulfilled. Let us examine assumption (ii). It is easy to show that $\partial f^0(x^0) = \{1\}$, $T_{p^0}(x^0) = \{1\}$ and $N_\Omega(x^0) = \{0\}$. If there exists $T_0 \subset T_{p^0}(x^0)$ such that $|T_0| < 1$ then $T_0 = \emptyset$. Hence

$$-\partial f^0(x^0) \cap \left[\operatorname{cone}\left(\bigcup_{t \in T_0} \partial g_t^0(x^0)\right) + N_\Omega(x^0)\right] = \emptyset,$$

and assumption (ii) is fulfilled. Applying Theorem 3.1 we conclude that $\mathcal{S}$ is pseudo-Lipschitz at (p^0, x^0).

The following examples show that the assertion of Theorem 3.1 may be false, if one of the assumptions (i) and (ii) is violate.

Example 3.2 For PCSI, let $T = \{1, 2, 3\} \cup [4, 5] \subset \mathbb{R}$, $P := \mathbf{CO}(\mathbb{R}^2) \times \mathbf{C}_1(\mathbb{R}^2 \times T)$ and $\Omega := \mathbb{R}^2_+$. Let $p^0 = (f^0, g^0) \in P$ be formulated by

$$f^0(x) = (x_1)^2 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,$$

$$g_t^0(x) = \begin{cases} -x_1, & \text{if } t = 1, 2, \\ -x_2, & \text{if } t = 3, \\ x_1 + x_2 - 1, & \text{if } t \in [4, 5], \ \forall x = (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

Let $x^0 := (0, 0) \in \mathcal{S}(p^0)$. We now check the assumptions of Theorem 3.1. Clearly, $\hat{x} = (\frac{1}{4}, \frac{1}{4})$ is a Slater element for p^0. However, assumption (ii) is violate. Indeed, we have $T_{p^0}(x^0) = \{1, 2, 3\}$, $T_0 = \{2\} \subset T_{p^0}(x^0)$ and $|T_0| = 1 < 2$. It is easily seen that $\partial f^0(x^0) = (0, 0)$, $\partial g_2^0(x^0) = \{(-1, 0)\}$ and $N_\Omega(x^0) = -\mathbb{R}_+^2$. It follows that $\text{cone}(\bigcup_{t \in T_0} \partial g_t^0(x^0)) = -\mathbb{R}_+ \times \{0\}$ and

$$-\partial f^0(x^0) \cap \left[\text{cone}\left(\bigcup_{t \in T_0} \partial g_t^0(x^0)\right) + N_\Omega(x^0)\right] = \{0\}.$$

We next examine the pseudo-Lipschitz property of $\mathcal{S}$ at (p^0, x^0). Let $\{p^k = (f^0, g^k)\}_{k=1}^\infty \subset P$ be a sequence such that

$$g_t^k(x) = \begin{cases} \frac{1}{2k} x_2 - x_1, & \text{if } t = 1, \\ -x_1, & \text{if } t = 2, \\ -x_2, & \text{if } t = 3, \\ x_1 + x_2 - 1, & \text{if } t \in [4, 5]. \end{cases}$$

We claim that $p^k = (f^0, g^k) \to p^0 = (f^0, g^0)$ as $k \to \infty$. Indeed, it is sufficient to show that $g^k \to g^0$ as $k \to \infty$. Let $\{K_m\}_{m=1}^\infty$ be a sequence of compact subset of $\mathbb{R}^2$ such that $K_m := m(\text{cl}\,\mathbb{B})$, where $\mathbb{B}$ stands for the open unit ball of $\mathbb{R}^2$. Then $\mathbb{R}^2 = \bigcup_{m=1}^\infty K_m$. Clearly,

$$\delta_m(g_t^k, g_t^0) := \max_{x \in K_m} |g_t^k(x) - g_t^0(x)| = \begin{cases} \frac{1}{2k} m, & \text{if } t = 1, \\ 0, & \text{if } t \in \{2, 3\} \cup [4, 5]. \end{cases}$$

Hence, $\sigma(g_t^k, g_t^0) = 0$ for $t \in \{2, 3\} \cup [4, 5]$, and for $t = 1$ we have

$$\sigma(g_1^k, g_1^0) := \sum_{m=1}^\infty \frac{1}{2^m} \frac{\sigma_m(g_1^k, g_1^0)}{1 + \sigma_m(g_1^k, g_1^0)} \le \frac{1}{2k} \sum_{m=1}^\infty \frac{1}{2^m} = \frac{1}{2k}.$$

Then $\sigma_\infty(g^k, g^0) := \max_{t \in T} \sigma(g_t^k, g_t^0) \le \frac{1}{2k}$ and so, $g^k \to g^0$ as $k \to \infty$.

We see that $\mathcal{S}$ is not pseudo-Lipschitz at (p^0, x^0). Indeed, taking $\bar{p}^k := (f^0, g^0)$, $\bar{x}^k = (1, 0) \in \mathcal{S}(\bar{p}^k)$ and $x^k \in \mathcal{S}(p^k) = \{(0, 0)\}$, we have

$$1 = d(\bar{x}^k, \mathcal{S}(p^k)) > \frac{1}{2} \ge k d(p^k, \bar{p}^k) \quad \forall k \ge 1.$$

Thus, $\mathcal{S}$ is not pseudo-Lipschitz at (p^0, x^0).

Example 3.3 For PCSI, let $T = [0,1] \cup \{2\}$, $P := \mathbf{CO}(\mathbb{R}) \times \mathbf{C}_1(\mathbb{R} \times T)$ and $\Omega := \mathbb{R}$. Let $p^0 = (f^0, g^0) \in P$ be defined by

$$f^0(x) = x + 1, \quad \forall x \in \mathbb{R},$$

$$g_t^0(x) = \begin{cases} -tx - t, & \text{if } t \in [0,1], \\ 0, & \text{if } t = 2, \ \forall x \in \mathbb{R}. \end{cases}$$

Then, $\mathcal{C}(p^0) = [-1, +\infty[$. Let $x^0 = -1$. We now check the assumptions of Theorem 3.1. Clearly, p^0 does not satisfy the Slater constraint qualification by $g_0^0(x) = 0$ for all $x \in \mathbb{R}$. Hence, assumption (i) is violate. Let us examine assumption (ii). It is easily seen that $\partial f^0(x^0) = \{1\}$, $T_{p^0}(x^0) = [0,1] \cup \{2\}$ and $N_\Omega(x^0) = \{0\}$. If there exists $T_0 \subset T_{p^0}(x^0)$ such that $|T_0| < 1$ then $T_0 = \emptyset$. Hence,

$$-\partial f^0(x^0) \cap \left[\text{cone}\left(\bigcup_{t \in T_0} \partial g_t^0(x^0)\right) + N_\Omega(x^0)\right] = \emptyset,$$

and so, assumption (ii) is fulfilled.

We next check the pseudo-Lipschitz property of $\mathcal{S}$ at (p^0, x^0). We have $\mathcal{S}(p^0) = \{-1\}$. Let $p^k = (f^0, g^k) \in P$,

$$g_t^k(x) = \begin{cases} -tx - \frac{k+1}{k}t + \frac{1}{k}, & \text{if } t \in [\frac{1}{k+1}, 1], \\ 0, & \text{if } t \in [0, \frac{1}{k+1}[\cup \{2\}. \end{cases}$$

We claim that $p^k \to p^0$ as $k \to \infty$. Indeed, it is sufficient to show that $g^k \to g^0$ as $k \to \infty$. Let $\{K_m\}_{m=1}^\infty$ be a sequence of compact subset of $\mathbb{R}$ such that $K_m := m[-1,1]$. Then $\mathbb{R} = \bigcup_{m=1}^\infty K_m$. Clearly,

$$\delta_m(g_t^k, g_t^0) := \max_{x \in K_m} |g_t^k(x) - g_t^0(x)| = \begin{cases} 0, & \text{if } t \in \{0, 2\}, \\ t(m+1), & \text{if } t \in]0, \frac{1}{k+1}[, \\ -\frac{1}{k}t + \frac{1}{k}, & \text{if } t \in [\frac{1}{k+1}, 1]. \end{cases}$$

Hence, $\sigma(g_t^k, g_t^0) = 0$ for $t \in \{0, 2\}$. For $t \in [\frac{1}{k+1}, 1]$ we have

$$\sigma(g_t^k, g_t^0) = \sum_{m=1}^\infty \frac{1}{2^m} \frac{\delta_m(g_t^k, g_t^0)}{1 + \delta_m(g_t^k, g_t^0)} = \left(-\frac{1}{k}t + \frac{1}{k}\right) \sum_{m=1}^\infty \frac{1}{2^m}.$$

This implies $\sigma(g_t^k, g_t^0) = -\frac{1}{k}t + \frac{1}{k}$ for all $t \in [\frac{1}{k+1}, 1]$. For $t \in]0, \frac{1}{k+1}[$ we have

$$\sigma(g_t^k, g_t^0) = \sum_{m=1}^\infty \frac{1}{2^m} \frac{t(m+1)}{1 + t(m+1)} = t \sum_{m=1}^\infty \frac{1}{2^m} \frac{(m+1)}{\frac{1}{t} + (m+1)} \le t \sum_{m=1}^\infty \frac{1}{2^m} = t.$$

Therefore,

$$\sigma_\infty(g^k, g^0) := \max_{t \in T} \sigma(g_t^k, g_t^0) \le \frac{1}{k+1},$$

and so, $g^k \to g^0$ as $k \to \infty$. This implies that $p^k \to p^0$ as $k \to \infty$. It is a simple matter to check that $\mathcal{S}(p^k) = \{0\}$ for every $k \geq 1$. Thus, $\mathcal{S}$ is not pseudo-Lipschitz at (p^0, x^0).

We now consider a special case of PCSI which has the form

$$(\mathrm{CSI})_{(c,b)}: \quad \min f(x) + c^T x \quad \text{subject to } x \in \mathbb{R}^n,\ g_t(x) \leq b_t,\ t \in T,$$

where $c \in \mathbb{R}^n$, c^T denotes the transpose of c, $f: \mathbb{R}^n \to \mathbb{R}$ and $g_t: \mathbb{R}^n \to \mathbb{R}$ $(t \in T)$ are given convex functions such that $(x,t) \mapsto g_t(x)$ is continuous on $\mathbb{R}^n \times T$, and $b \in \mathbf{C}_0(T)$-the set of all continuous functions on T. The set of feasible points of $(\mathrm{CSI})_{(c,b)}$ is denoted by $\Gamma(c,b)$. $\mathcal{S}(c,b)$ stands for the set of all solutions of $(\mathrm{CSI})_{(c,b)}$. The set of active constraints at $x \in \Gamma(c,b)$ is given by

$$T_{(c,b)}(x) := \{t \in T \mid g_t(x) = b_t\}.$$

The following corollary is an immediate consequence of Theorem 3.1 by taking

$$\tilde{f}(x) = f(x) + c^T x, \quad \tilde{g}(x,t) = g_t(x) - b_t, \quad \text{and} \quad \Omega = \mathbb{R}^n.$$

Corollary 3.1 [5, Theorem 10] *For* $(\mathrm{CSI})_{(c,b)}$, *let* $c^0 \in \mathbb{R}^n$, $b^0 \in \mathbf{C}_0(T)$ *and* $x^0 \in \mathcal{S}(c^0, b^0)$. *Suppose that the following conditions hold*:

(i) (c^0, b^0) *satisfies the Slater constraint qualification*;
(ii) *There is no* $T_0 \subset T_{(c^0,b^0)}(x^0)$ *with* $|T_0| < n$ *satisfying*

$$-\big(c^0 + \partial f^0(x^0)\big) \cap \operatorname{cone}\Bigg(\bigcup_{t \in T_0} \partial g_t^0(x^0)\Bigg) \neq \emptyset.$$

Then $\mathcal{S}$ *is pseudo-Lipschitz at* $((c^0, b^0), x^0)$.

4 Application to Linear Semi-Infinite Optimization Problems

In this section we establish sufficient conditions for pseudo-Lipschitz property of the solution mapping of parametric linear semi-infinite optimization problems at the nominal parameter.

Let T be a nonempty compact metric space and let $\mathbf{C}_0(T)$ and $\mathbf{C}_0(T, \mathbb{R}^n)$ be, respectively, the set of all continuous mappings

$$b: T \to \mathbb{R} \quad \text{and} \quad a: T \to \mathbb{R}^n$$

normed by

$$\|b\|_\infty := \sup_{t \in T} |b(t)| \text{ and } \|a\|_\infty := \sup_{t \in T} \|a(t)\|,$$

where $\|\cdot\|$ denotes Euclidean norm in $\mathbb{R}^n$.

For every triple of parameters $p = (c, a, b) \in P := \mathbb{R}^n \times \mathbf{C}_0(T, \mathbb{R}^n) \times \mathbf{C}_0(T)$ we consider the linear semi-infinite problem

$$(\mathrm{LSI})_p \quad \min\langle c, x\rangle \quad \text{subject to } x \in \mathcal{C}(p),$$

where $\mathcal{C}(p) := \{x \in \Omega | \langle a(t), x\rangle \leq b(t) \text{ for every } t \in T\}$ and Ω is a nonempty closed convex subset of $\mathbb{R}^n$.

Theorem 4.1 *Let $p^0 = (c_0, a_0, b_0) \in P$ and $x^0 \in \mathcal{S}(p^0)$. Suppose that the following conditions hold*:

(i) *p^0 satisfies the Slater constraint qualification*;
(ii) *There is no $T_0 \subset T_{p^0}(x^0)$ with $|T_0| < n$ satisfying*

$$-c_0 \in \mathrm{cone}(\{a_0(t) \mid t \in T_0\}) + N_\Omega(x^0).$$

Then $\mathcal{S}$ is pseudo-Lipschitz at (p^0, x^0).

Proof The assertion of the theorem follows by the same method as in the proof of Theorem 3.1 with $d_\infty(\cdot)$ replaced by $\|\cdot\|_\infty$ as well noting that $\langle v_t - \bar{v}_t, x\rangle \leq \sqrt{n}\|v - \bar{v}\|_\infty\|x\|$ for every $v,\ \bar{v} \in \mathbf{C}_0(T, \mathbb{R}^n)$, $x \in \mathbb{R}^n$ and for every $t \in T$. □

Finally, in the case that $\Omega = \mathbb{R}^n$ and p just depends on the parameters c and b, i.e., the problems under consideration become the linear semi-infinite optimization problems under continuous perturbations of the right-hand side of the constraints and linear perturbations of the objective function. For this class of problems, the pseudo-Lipschitz property holds, if and only if both conditions (i) and (ii) of Theorem 4.1 hold at the nominal parameter, as shown in [5, Theorem 16]. However, we do not known at present how to proceed with the case that p depends on the triple of parameters c, a and b.

5 Concluding Remarks

We have established sufficient conditions for the pseudo-Lipschitz property of the solution map of parametric convex semi-infinite optimization problem (PCSI) under convex functional perturbations of both objective function and constraint set. Moreover, examples are provided to illustrate the obtained results. An application to parametric linear semi-infinite optimization problems is also given.

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