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Iterative Methods for Triple Hierarchical Variational Inequalities in Hilbert Spaces

Lu-Chuan Ceng · Qamrul Hasan Ansari · Jen-Chih Yao

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Abstract In this paper, we consider a variational inequality with a variational inequality constraint over a set of fixed points of a nonexpansive mapping called triple hierarchical variational inequality. We propose two iterative methods, one is implicit and another one is explicit, to compute the approximate solutions of our problem. We present an example of our problem. The convergence analysis of the sequences generated by the proposed methods is also studied.

Keywords Triple hierarchical variational inequalities · Iterative methods · Implicit schemes · Explicit schemes · Fixed points · Nonexpansive mappings

1 Introduction

In the last two decades, the bilevel programming problems have been extensively studied in the literature because of their applications in mechanics and network design; see, for example, [1, 2] and the references therein. In the bilevel programming problems, the constraint set is a solution set of another problem, namely, optimization

L.-C. Ceng
Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
e-mail: zenglc@hotmail.com

L.-C. Ceng
Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, China

Q.H. Ansari (✉)
Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
e-mail: qhansari@gmail.com

J.-C. Yao (✉)
Center for General Education, Kaohsiung Medical University, Kaohsiung 80708, Taiwan
e-mail: yaojc@cc.kmu.edu.tw

problem, variational inequality, fixed point problem, etc. For details and applications of variational inequalities, we refer to [3–7] and the references therein. A variational inequality over the set of fixed points of a mapping is called a hierarchical variational inequality, also known as a hierarchical fixed point problem. It is worth to mention that the signal recovery [8], beamforming [9] and power control [10] can be written in the form of a hierarchical variational inequality. Such a problem is also a bilevel problem, in which we find a solution of the first problem subject to the condition that its solution also be a fixed point of a mapping. For further details on hierarchical fixed point problems and their applications, we refer to [8, 11–24] and the references therein. The solution presented in [15, 17] is not always unique. So, it must be found from among candidate solutions. Therefore, it would be reasonable to identify the unique minimizer of an appropriate objective function over the hierarchical fixed point constraint. To find a such unique minimizer, Iiduka [25, 26] introduced a triple hierarchical variational inequality (IHVI) and proposed an iterative method to find its solution.

The present paper is in twofold. First, we combine the regularization method, the hybrid steepest-descent method, and the projection method to propose an implicit scheme that generates a net in an implicit way, and study its strong convergence to a unique solution of the THVI. Second, we introduce an explicit scheme that generates a sequence via an iterative algorithm and prove that this sequence converges strongly to a unique solution of the THVI. In particular cases, the results presented in this paper reduce to the corresponding results in [24].

2 Formulations and Preliminaries

We write to indicate that the sequence $\{x_n\}$ converges weakly to x.

Throughout the paper, unless otherwise specified, we denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. We assume that H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We also assume that C be a nonempty, closed, and convex subset of H and $f : C \to H$ be a (possibly nonself) ρ-contraction mapping with contractivity constant $\rho \in [0, 1[$.

Let $F : C \to H$ be κ-Lipschitzian and η-strongly monotone, where $\kappa > 0$ and $\eta > 0$ are constants, that is,

$$\|Fx - Fy\| \le \kappa \|x - y\|, \quad \forall x, y \in C \tag{1}$$

and

$$\langle Fx - Fy, x - y\rangle \ge \eta \|x - y\|^2, \quad \forall x, y \in C. \tag{2}$$

Let $T : H \to H$ be a nonexpansive mapping such that the set of fixed points of T, $\mathrm{Fix}(T) = \{x \in H : Tx = x\} \ne \emptyset$. In 2001, Yamada [20] introduced the so-called hybrid steepest-descent method for solving hierarchical variational inequality (in short, HVI) of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\left\langle Fx^*, x - x^*\right\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3}$$

This method generates a sequence $\{x_n\}$ via the following iterative algorithm:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad \forall n \geq 0, \tag{4}$$

where $0 < \mu < 2\eta/\kappa^2$, the initial guess $x_0 \in H$ is arbitrary and the sequence $\{\lambda_n\}$ in $]0, 1[$ satisfies the following conditions:

$$\lambda_n \to 0, \quad \sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

A key fact in Yamada's argument is that, for small enough $\lambda > 0$, the mapping

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H$$

is a contraction due to the κ-Lipschitz continuity and η-strong monotonicity of F. However, this method cannot be applied to the following HVI considered in [24]: find $x^* \in \operatorname{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T), \tag{5}$$

where $T, S : C \to C$ are two nonexpansive mappings and $\operatorname{Fix}(T) = \{x \in C : Tx = x\}$ is the set of fixed points of T. Let Ω denote the set of solutions of HVI (5) and assume that Ω is nonempty; consequently, the metric projection P_Ω is well defined. It is interesting to find the minimum-norm solution x^* of the HVI (5) which exists uniquely and is exactly the nearest point projection of the origin to Ω, that is, $x^* = P_\Omega(0)$. Alternatively, x^* is the unique solution of the quadratic minimization problem:

$$\|x^*\|^2 = \min\{\|x\|^2 : x \in \Omega\}. \tag{6}$$

Yao et al. [24] used the contractions to regularize the nonexpansive mapping S to propose an implicit scheme that generates a net $\{x_t\}_{t\in]0,1[}$ in an implicit way and proved the strong convergence of $\{x_t\}$ to a minimum-norm solution x^* of the HVI (5). They also introduced an explicit scheme that generates a sequence $\{x_n\}$ via an iterative algorithm and proved that this sequence converges strongly to the same minimum-norm solution x^* of the HVI (5). Recently, Mainge and Moudafi [15] and Lu et al. [27] introduced a hybrid iterative method and a regularization method, respectively, for solving HVI (5).

Very recently, Iiduka [25, 26] considered a variational inequality with a variational inequality constraint over the set of fixed points of a nonexpansive mapping. Since this problem has a triple structure in contrast with bilevel programming problems or hierarchical constrained optimization problems or hierarchical fixed point problem, it is referred as triple hierarchical constrained optimization problem (THCOP). He presented some examples of THCOP and developed iterative algorithms to find the solution of such a problem. The convergence analysis of the proposed algorithms is also studied in [25, 26]. Since the original problem is a variational inequality, in this paper, we call it triple hierarchical variational inequality (THVI).

Let $S, T : C \to C$ be two nonexpansive mappings with $\operatorname{Fix}(T) \neq \emptyset$. Let

$$0 < \mu < 2\eta/\kappa^2 \quad \text{and} \quad 0 < \gamma \leq \tau,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$. We consider the following triple hierarchical variational inequality (for short, THVI): find $x^* \in \Xi$ such that

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi, \tag{7}$$

where Ξ denotes the solution set of the following hierarchical variational inequality: find $z^* \in \mathrm{Fix}(T)$ such that

$$\langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T); \tag{8}$$

where we assume that the solution set of HVI (8) is nonempty.

We present an example of a THVI. It also shows the relevance of generalization of double hierarchical variational inequality to triple hierarchical variational inequality.

Example 2.1 Let $H = \mathbb{R}^2$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ are defined by

$$\langle x, y \rangle = ac + bd \quad \text{and} \quad \|x\| = \sqrt{a^2+b^2}$$

for all $x, y \in \mathbb{R}^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{x \in \mathbb{R}^2 : \|x\| \leq \sqrt{2}\}$. Clearly, C is a nonempty, bounded, and closed convex subset of $\mathbb{R}^2$. Let f be a 2×2 positive semidefinite matrix such that $0 < \|f\| < 1$, for instance, putting $f = \left\{ \begin{smallmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{smallmatrix} \right\}$, we know that $\|f\| = \frac{2}{3}$ and $f : C \to C$ is a ρ-contraction with contractivity constant $\rho = \frac{2}{3}$. Let $F = \frac{1}{2}I = \left\{ \begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{smallmatrix} \right\}$. Then $F : C \to C$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa = \frac{1}{2}$ and $\eta = \frac{1}{2}$, respectively. Take $\mu = 2$ and $\gamma = 1$ such that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$ where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} = 1$. Let T and S be two 2×2 positive definite matrices such that $\|T\| = \|S\| = 1$, for instance, putting $T = \left\{ \begin{smallmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{smallmatrix} \right\}$ and $S = \left\{ \begin{smallmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} \end{smallmatrix} \right\}$, we know that $\|T\| = \|S\| = 1$ and that $S, T : C \to C$ are two nonexpansive mappings with $\mathrm{Fix}(T) = \{(a, a) : |a| \leq 1\} \neq \emptyset$.

We observe that the solution set Ξ of the HVI is the following:

$$\begin{aligned} \Xi &= \{z^* \in \mathrm{Fix}(T) : \langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)\} \\ &= \{z^* \in \mathrm{Fix}(T) : \langle (I - S)z^*, z - z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)\} \\ &= \{z^* \in \mathrm{Fix}(T) : \langle 0, z - z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)\} \\ &= \mathrm{Fix}(T) = \{(a, a) : |a| \leq 1\} \neq \emptyset. \end{aligned}$$

Further, it is easy to see that there exists a unique solution $x^* = (0, 0)$ to the following THVI: find $x^* \in \Xi$ $(= \mathrm{Fix}(T))$ such that

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi.$$

$$(\text{Namely}, \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi.)$$

Now, we present some known results and definitions which will be used in the sequel.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Lemma 2.1 *For any given $x \in H$ and $z \in C$.*

(i) $z = P_C x$ *if and only if* $\langle x - z, y - z\rangle \le 0$, *for all* $y \in C$.
(ii) $z = P_C x$ *if and only if* $\|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, *for all* $y \in C$.
(iii) *For all* $x, y \in H$,

$$\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2.$$

Consequently, P_C *is nonexpansive and monotone.*

Below we gather some basic facts that are needed in the sequel.

Lemma 2.2 [28, Demiclosedness Principle] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in$* $\mathrm{Fix}(T)$.

The following lemma plays a key role in proving strong convergence of the sequences generated by our algorithms.

Lemma 2.3 [29, Lemma 2.1] *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \le (1 - \gamma_n)s_n + \gamma_n \delta_n, \quad \forall n \ge 0,$$

where $\{\gamma_n\}$, $\{\delta_n\}$ are sequences of real numbers such that

(i) $\{\gamma_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \gamma_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - \gamma_k) = 0;$$

(ii) $\limsup_{n\to\infty} \delta_n \le 0$, *or*
(ii)′ $\sum_{n=0}^{\infty} \gamma_n \delta_n$ *is convergent.*

Then, $\lim_{n\to\infty} s_n = 0$.

The following lemma is not hard to prove.

Lemma 2.4 [30] *Let $f : C \to H$ be a ρ-contraction with $\rho \in [0, 1[$ and $T : C \to C$ be a nonexpansive mapping. Then*

(i) *$I - f$ is $(1-\rho)$-strongly monotone*:

$$\langle (I-f)x - (I-f)y, x-y\rangle \geq (1-\rho)\|x-y\|^2, \quad \forall x, y \in C;$$

(ii) *$I - T$ is monotone*:

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq 0, \quad \forall x, y \in C.$$

The following fact is straightforward but useful.

Lemma 2.5 *There holds the following inequality in an inner product space X*:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Lemma 2.6 [29, Lemma 3.1] *Let λ be a number in $]0, 1]$ and let $\mu > 0$. Let $F : C \to H$ be an operator on C such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\left\|T^\lambda x - T^\lambda y\right\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in]0, 1]$.

Remark 2.1 Put $F = \frac{1}{2}I$, where I is the identity operator of H. Then we have $\mu < 2\eta/\kappa^2 = 4$. Also, put $\mu = 2$. Then it is easy to see that $\kappa = \eta = \frac{1}{2}$ and

$$\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} = 1 - \sqrt{1 - 2\left(2\cdot\frac{1}{2} - 2\left(\frac{1}{2}\right)^2\right)} = 1.$$

In particular, whenever $\lambda > 0$, we have $T^\lambda x := Tx - \lambda\mu F(Tx) = (1-\lambda)Tx$.

3 An Implicit Scheme

Our method depends on a regularization on the nonexpansive mapping S in HVI (8). More precisely, we use the (possibly nonself) contraction f in THVI (7) and a real number $t \in\ ]0, 1[$ to construct the contraction $tf + (1-t)S$ such that S is regularized. To get involved with the second nonexpansive mapping T in HVI (8), we make a linear combination $s\gamma(tf + (1-t)S) + (I - s\mu F)T$ of the regularization $tf + (1-t)S$ and the steepest-descent technique $(I - s\mu F)T$, where $s \in\ ]0, 1[$. This mapping is still a contraction. However, it may be a nonself-mapping on C since F and f can be nonself. So, in order to remain in C, we apply the nearest point projection P_C from H onto C. That is, we consider the contraction mapping

$$x \mapsto f_{s,t}(x) := P_C[s\gamma(tf + (1-t)S) + (I - s\mu F)T]x.$$

Note that this contraction is a self-mapping on C. It is easy to find that the contraction coefficient of $f_{s,t}$ is $1-(1-\rho)\gamma st$. Indeed, in terms of Lemma 2.6 we obtain that for each $x, y \in C$

$$\begin{aligned}
&\|f_{s,t}(x) - f_{s,t}(y)\| \\
&\quad = \|P_C[s\gamma(tf + (1-t)S) + (I - s\mu F)T]x - P_C[s\gamma(tf + (1-t)S) \\
&\qquad + (I - s\mu F)T]y\| \\
&\quad \le \|[s\gamma(tf + (1-t)S) + (I - s\mu F)T]x - [s\gamma(tf + (1-t)S) \\
&\qquad + (I - s\mu F)T]y\| \\
&\quad \le s\gamma\|(tf + (1-t)S)x - (tf + (1-t)S)y\| + \|(I - s\mu F)Tx \\
&\qquad - (I - s\mu F)Ty\| \\
&\quad \le s\gamma\|tf(x) + (1-t)Sx - tf(y) - (1-t)Sy\| + (1 - s\tau)\|x - y\| \\
&\quad \le s\gamma[t\|f(x) - f(y)\| + (1-t)\|Sx - Sy\|] + (1 - s\tau)\|x - y\| \\
&\quad \le s\gamma[t\rho\|x - y\| + (1-t)\|x - y\|] + (1 - s\tau)\|x - y\| \\
&\quad = s\gamma(1 - t(1-\rho))\|x - y\| + (1 - s\tau)\|x - y\| \\
&\quad = \{1 - s[\tau - \gamma(1 - t(1-\rho))]\}\|x - y\| \\
&\quad \le (1 - st\gamma(1-\rho))\|x - y\|
\end{aligned}$$

due to $0 < \gamma \le \tau$. Since $0 < \gamma \le \tau \le 1,\ 0 \le \rho < 1$ and $0 < s, t < 1$, we get $st\gamma(1-\rho) < \gamma(1-\rho) \le 1$. This implies that the contraction coefficient of $f_{s,t}$ is $1-(1-\rho)\gamma st$. Hence, by the Banach contraction principle, $f_{s,t}$ has a unique fixed point which is denoted by $x_{s,t} \in C$, that is, $x_{s,t}$ is the unique solution in C of the fixed-point equation

$$x_{s,t} = P_C[s\gamma(tf(x_{s,t}) + (1-t)Sx_{s,t}) + (I - s\mu F)Tx_{s,t}]. \tag{9}$$

Additionally, if we take $f = 0$, then (9) reduces to

$$x_{s,t} = P_C[\gamma s(1-t)Sx_{s,t} + (I - s\mu F)Tx_{s,t}]. \tag{10}$$

In particular, whenever $\mu = 2,\ F = \frac{1}{2}I$ and $\gamma = \tau = 1$, the implicit schemes (9) and (10) reduce to the following implicit schemes, respectively:

$$x_{s,t} = P_C[s(tf(x_{s,t}) + (1-t)Sx_{s,t}) + (1-s)Tx_{s,t}] \tag{11}$$

and

$$x_{s,t} = P_C[s(1-t)Sx_{s,t} + (1-s)Tx_{s,t}]. \tag{12}$$

Below is the first result of this paper which displays the behavior of the net $\{x_{s,t}\}$ as $s \to 0$ and $t \to 0$ successively.

Theorem 3.1 *Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ and $\eta > 0$, respectively, $f : C \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1[$ and $S, T : C \to C$ be two nonexpansive mappings with* $\mathrm{Fix}(T) \neq \emptyset$. *Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \le \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)}$. Suppose that the solution set Ξ of HVI (8) be nonempty. For each $(s,t) \in\ ]0,1[\times]0,1[$, let $x_{s,t}$ be defined implicitly by (9). Then, for each fixed $t \in\ ]0,1[$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in \mathrm{Fix}(T)$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to a unique solution $x^* \in \Xi$ of the THVI (7). Moreover, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$ such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.*

In particular, if we take $f = 0$ and if $x_{s,t}$ is defined by the implicit scheme (10), then the iterated limit in the norm topology

$$s - \lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is a unique solution x^* of the variational inequality (in short, VI), which consists in finding $x^* \in \Xi$ such that

$$\langle Fx^*, x - x^*\rangle \ge 0, \quad \forall x \in \Xi. \tag{13}$$

Furthermore, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Proof We first show that $\{x_{s,t}\}$ is bounded. Indeed, take any $z \in \mathrm{Fix}(T)$. Observe that for all $s, t \in]0,1[$

$$\begin{aligned} &s\gamma(tf(x_{s,t}) + (1-t)Sx_{s,t}) + (I - s\mu F)Tx_{s,t} - z \\ &\quad = [(I - s\mu F)Tx_{s,t} - (I - s\mu F)z] + st\gamma(f(x_{s,t}) - f(z)) \\ &\qquad + s(1-t)\gamma(Sx_{s,t} - Sz) + st(\gamma f - \mu F)z + s(1-t)(\gamma S - \mu F)z. \end{aligned}$$

Noticing $0 < \gamma \le \tau$ and utilizing Lemma 2.6, we have

$$\begin{aligned} \|x_{s,t} - z\| &= \|P_C[s\gamma(tf(x_{s,t}) + (1-t)Sx_{s,t}) + (I - s\mu F)Tx_{s,t}] - z\| \\ &\le \|s\gamma(tf(x_{s,t}) + (1-t)Sx_{s,t}) + (I - s\mu F)Tx_{s,t} - z\| \\ &= \|[(I - s\mu F)Tx_{s,t} - (I - s\mu F)z] + st\gamma(f(x_{s,t}) - f(z)) \\ &\quad + s(1-t)\gamma(Sx_{s,t} - Sz) + st(\gamma f - \mu F)z + s(1-t)(\gamma S - \mu F)z\| \end{aligned}$$

$$
\begin{aligned}
&\le \|(I-s\mu F)Tx_{s,t}-(I-s\mu F)z\|+st\gamma\|f(x_{s,t})-f(z)\|\\
&\quad +s(1-t)\gamma\|Sx_{s,t}-Sz\|+st\|(\gamma f-\mu F)z\|\\
&\quad +s(1-t)\|(\gamma S-\mu F)z\|\\
&\le (1-s\tau)\|x_{s,t}-z\|+st\gamma\rho\|x_{s,t}-z\|+s(1-t)\gamma\|x_{s,t}-z\|\\
&\quad +st\|(\gamma f-\mu F)z\|+s(1-t)\|(\gamma S-\mu F)z\|\\
&\le [1-s\tau+st\gamma\rho+s(1-t)\gamma]\|x_{s,t}-z\|\\
&\quad +(st+s(1-t))\max\{\|(\gamma f-\mu F)z\|,\|(\gamma S-\mu F)z\|\}\\
&\le (1-st\gamma(1-\rho))\|x_{s,t}-z\|+s\max\{\|(\gamma f-\mu F)z\|,\|(\gamma S-\mu F)z\|\}.
\end{aligned}
$$

This implies that

$$
\|x_{s,t}-z\|\le \frac{1}{t\gamma(1-\rho)}\max\{\|(\gamma f-\mu F)z\|,\|(\gamma S-\mu F)z\|\}.
$$

It follows that for each fixed $t\in\ ]0,1[$, $\{x_{s,t}\}$ is bounded and so are the nets $\{Tx_{s,t}\}$, $\{Sx_{s,t}\}$, $\{f(x_{s,t})\}$, $\{Fx_{s,t}\}$ and $\{FTx_{s,t}\}$. Since $x_{s,t}\in C$ and also $Tx_{s,t}\in C$, we get

$$
\begin{aligned}
\|x_{s,t}-Tx_{s,t}\| &= \|P_C[s\gamma(tf(x_{s,t})+(1-t)Sx_{s,t})+(I-s\mu F)Tx_{s,t}]-P_CTx_{s,t}\|\\
&\le s\|\gamma(tf(x_{s,t})+(1-t)Sx_{s,t})-\mu FTx_{s,t}\|\\
&\to 0 \quad \text{as } s\to 0 \text{ for each fixed } t\in\ ]0,1[. \qquad (14)
\end{aligned}
$$

We show that for each fixed $t\in\ ]0,1[$, the net $\{x_{s,t}\}$ is relatively norm-compact as $s\to 0$. Set

$$
y_{s,t}=s\gamma(tf(x_{s,t})+(1-t)Sx_{s,t})+(I-s\mu F)Tx_{s,t}.
$$

Then we have $x_{s,t}=P_Cy_{s,t}$, and for any $w\in \mathrm{Fix}(T)$,

$$
\begin{aligned}
x_{s,t}-w &= P_Cy_{s,t}-y_{s,t}+y_{s,t}-w\\
&= P_Cy_{s,t}-y_{s,t}+s\gamma(tf(x_{s,t})+(1-t)Sx_{s,t})+(I-s\mu F)Tx_{s,t}-w\\
&= P_Cy_{s,t}-y_{s,t}+[(I-s\mu F)Tx_{s,t}-(I-s\mu F)w]\\
&\quad +st\gamma(f(x_{s,t})-f(w))+s(1-t)\gamma(Sx_{s,t}-Sw)+st(\gamma f-\mu F)w\\
&\quad +s(1-t)(\gamma S-\mu F)w. \qquad (15)
\end{aligned}
$$

Since P_C is the metric projection from H onto C, we have

$$
\langle P_Cy_{s,t}-y_{s,t},P_Cy_{s,t}-w\rangle\le 0.
$$

It follows from (15) that

$$\begin{aligned}\|x_{s,t}-w\|^2 &= \langle P_C y_{s,t}-y_{s,t}, P_C y_{s,t}-w\rangle + \langle (I-s\mu F)Tx_{s,t} \\ &\quad -(I-s\mu F)w, x_{s,t}-w\rangle + st\gamma\langle f(x_{s,t})-f(w), x_{s,t}-w\rangle \\ &\quad + s(1-t)\gamma\langle Sx_{s,t}-Sw, x_{s,t}-w\rangle + st\langle(\gamma f-\mu F)w, x_{s,t}-w\rangle \\ &\quad + s(1-t)\langle(\gamma S-\mu F)w, x_{s,t}-w\rangle \\ &\le [1-s\tau+st\gamma\rho+s(1-t)\gamma]\|x_{s,t}-w\|^2 \\ &\quad + st\langle(\gamma f-\mu F)w, x_{s,t}-w\rangle + s(1-t)\langle(\gamma S-\mu F)w, x_{s,t}-w\rangle \\ &\le (1-st\gamma(1-\rho))\|x_{s,t}-w\|^2 + st\langle(\gamma f-\mu F)w, x_{s,t}-w\rangle \\ &\quad + s(1-t)\langle(\gamma S-\mu F)w, x_{s,t}-w\rangle.\end{aligned}$$

It turns out that

$$\|x_{s,t}-w\|^2 \le \frac{1}{t\gamma(1-\rho)}\langle(t\gamma f+(1-t)\gamma S-\mu F)w, x_{s,t}-w\rangle, \quad \forall w\in \operatorname{Fix}(T). \tag{16}$$

Assume $\{s_n\}\subset\ ]0,1[$ is such that $s_n\to 0$ as $n\to\infty$. From (16), we obtain immediately that

$$\|x_{s_n,t}-w\|^2 \le \frac{1}{t\gamma(1-\rho)}\langle(t\gamma f+(1-t)\gamma S-\mu F)w, x_{s_n,t}-w\rangle, \quad \forall w\in \operatorname{Fix}(T). \tag{17}$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that $\{x_{s_n,t}\}$ converges weakly to a point $x_t\in C$. From (14), we get $\|x_{s_n,t}-Tx_{s_n,t}\|\to 0$. So, Lemma 2.2 implies that $x_t\in \operatorname{Fix}(T)$. We can then substitute x_t for w in (17) to get

$$\|x_{s_n,t}-x_t\|^2 \le \frac{1}{t\gamma(1-\rho)}\langle(t\gamma f+(1-t)\gamma S-\mu F)x_t, x_{s_n,t}-x_t\rangle.$$

Consequently, the weak convergence of $\{x_{s_n,t}\}$ to x_t actually implies that $x_{s_n,t}\to x_t$ strongly. This has proved the relative norm-compactness of the net $\{x_{s,t}\}$ as $s\to 0$.

Now, we return to (17) and take the limit as $n\to\infty$ to get

$$\|x_t-w\|^2 \le \frac{1}{t\gamma(1-\rho)}\langle(t\gamma f+(1-t)\gamma S-\mu F)w, x_t-w\rangle, \quad \forall w\in \operatorname{Fix}(T).$$

In particular, x_t solves the HVI of finding $x_t\in \operatorname{Fix}(T)$ such that

$$\langle(t\gamma f+(1-t)\gamma S-\mu F)w, x_t-w\rangle\ge 0, \quad \forall w\in \operatorname{Fix}(T),$$

that is,

$$\langle(\mu F-t\gamma f-(1-t)\gamma S)w, w-x_t\rangle\ge 0, \quad \forall w\in \operatorname{Fix}(T).$$

Let us show that the mapping $(\mu F - t\gamma f - (1-t)\gamma S)$ is monotone. Indeed, we observe that for each $x, y \in C$,

$$\begin{aligned}
&\langle(\mu F - t\gamma f - (1-t)\gamma S)x - (\mu F - t\gamma f - (1-t)\gamma S)y, x-y\rangle \\
&\quad = \mu\langle Fx - Fy, x-y\rangle - t\gamma\langle f(x) - f(y), x-y\rangle - (1-t)\gamma\langle Sx - Sy, x-y\rangle \\
&\quad \ge \mu\eta\|x-y\|^2 - t\gamma\rho\|x-y\|^2 - (1-t)\gamma\|x-y\|^2 \\
&\quad = [(\mu\eta - \gamma) + t\gamma(1-\rho)]\|x-y\|^2.
\end{aligned}$$

Noticing the inequality $\mu\eta \ge \tau$ (the argument can be seen in the sequel), we conclude from $0 < \gamma \le \tau$ and $\rho \in [0,1[$ that

$$(\mu\eta - \gamma) + t\gamma(1-\rho) \ge (\mu\eta - \tau) + t\gamma(1-\rho) > 0.$$

This shows that the mapping $(\mu F - t\gamma f - (1-t)\gamma S)$ is strongly monotone, and hence, monotone. It is easy to see that the mapping $(\mu F - t\gamma f - (1-t)\gamma S)$ is Lipschitz continuous. It is well known that the set $\mathrm{Fix}(T) \neq \emptyset$ is closed and convex; see, for example, [28]. Then, by applying the well-known Minty lemma (see [5]) for the operator $(\mu F - t\gamma f - (1-t)\gamma S)$ and the set $\mathrm{Fix}(T)$, we conclude that x_t solves the Minty variational inequality of finding $x_t \in \mathrm{Fix}(T)$ such that

$$\langle(t\gamma f + (1-t)\gamma S - \mu F)x_t, x_t - w\rangle \ge 0, \quad \forall w \in \mathrm{Fix}(T). \tag{18}$$

Notice that (18) is equivalent to the fact that $x_t = P_{\mathrm{Fix}(T)}(I - \mu F + t\gamma f + (1-t)\gamma S)x_t$. That is, x_t is a unique fixed point in $\mathrm{Fix}(T)$ of the contraction

$$P_{\mathrm{Fix}(T)}(I - \mu F + t\gamma f + (1-t)\gamma S).$$

Obviously, this is sufficient to conclude that the entire net $\{x_{s,t}\}$ converges in norm to x_t as $s \to 0$.

Next, we show that as $t \to 0$, the net $\{x_t\}$ converges strongly to x^* which is a unique solution of the HVI (8).

In (18), we take any $y \in \Xi$ to derive

$$\langle(t\gamma f + (1-t)\gamma S - \mu F)x_t, x_t - y\rangle \ge 0. \tag{19}$$

Note that $0 < \gamma \le \tau$ and

$$\begin{aligned}
\mu\eta \ge \tau \quad &\Leftrightarrow \quad \mu\eta \ge 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\
&\Leftrightarrow \quad \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \ge 1 - \mu\eta \\
&\Leftrightarrow \quad 1 - 2\mu\eta + \mu^2\kappa^2 \ge 1 - 2\mu\eta + \mu^2\eta^2 \\
&\Leftrightarrow \quad \kappa^2 \ge \eta^2 \\
&\Leftrightarrow \quad \kappa \ge \eta.
\end{aligned}$$

It is clear that

$$\langle (\mu F - \gamma S)x - (\mu F - \gamma S)y, x - y\rangle \geq (\mu\eta - \gamma)\|x - y\|^2, \quad \forall x, y \in C.$$

Hence, it follows from $0 < \gamma \leq \tau \leq \mu\eta$ that $\mu F - \gamma S$ is monotone. Thus, we have

$$\langle \gamma S x_t - \mu F x_t, x_t - y\rangle \leq \langle \gamma S y - \mu F y, x_t - y\rangle \leq 0. \tag{20}$$

It follows from (19) and (20) that

$$\langle (\gamma f - \mu F)x_t, x_t - y\rangle \geq 0, \quad \forall y \in \Xi. \tag{21}$$

Hence,

$$\begin{aligned}
\mu\eta\|x_t - y\|^2 &\leq \mu\langle F x_t - F y, x_t - y\rangle \\
&\leq \langle \mu F y - \gamma f(x_t), y - x_t\rangle \\
&= \langle (\mu F - \gamma f)y, y - x_t\rangle + \gamma\langle f(y) - f(x_t), y - x_t\rangle \\
&\leq \langle (\mu F - \gamma f)y, y - x_t\rangle + \gamma\rho\|y - x_t\|^2.
\end{aligned}$$

Therefore,

$$\|x_t - y\|^2 \leq \frac{1}{\mu\eta - \gamma\rho}\langle (\mu F - \gamma f)y, y - x_t\rangle, \quad \forall y \in \Xi. \tag{22}$$

In particular,

$$\|x_t - y\| \leq \frac{1}{\mu\eta - \gamma\rho}\|(\mu F - \gamma f)y\|, \quad \forall t \in\ ]0, 1[.$$

which implies that $\{x_t\}$ is bounded.

Next, let us show that $\omega_w(\{x_t\}) \subset \Xi$; namely, if $\{t_n\}$ is a null sequence in $]0, 1[$ such that $x_{t_n} \to x'$ weakly as $n \to \infty$, then $x' \in \Xi$. To see this, we use (18) to get

$$\langle (\mu F - \gamma S)x_t, w - x_t\rangle \geq \frac{t}{1 - t}\langle (\mu F - \gamma f)x_t, w - x_t\rangle, \quad \forall w \in \mathrm{Fix}(T).$$

However, since $\mu F - \gamma S$ is monotone,

$$\langle (\mu F - \gamma S)w, w - x_t\rangle \geq \langle (\mu F - \gamma S)x_t, w - x_t\rangle.$$

Combining the last two relations yields

$$\langle (\mu F - \gamma S)w, w - x_t\rangle \geq \frac{t}{1 - t}\langle (\mu F - \gamma f)x_t, w - x_t\rangle, \quad \forall w \in \mathrm{Fix}(T). \tag{23}$$

Letting $t = t_n \to 0$ as $n \to \infty$ in (23), we get

$$\left\langle (\mu F - \gamma S)w, w - x'\right\rangle \geq 0, \quad \forall w \in \mathrm{Fix}(T). \tag{24}$$

Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\mathrm{Fix}(T)$ is nonempty, closed and convex, by applying Minty lemma [5] on the set $\mathrm{Fix}(T)$ and on the operator $\mu F - \gamma S$, the inequality (24) is equivalent to

$$\big\langle (\mu F - \gamma S)x', w - x'\big\rangle \geq 0, \quad \forall w \in \mathrm{Fix}(T).$$

Namely, x' is a solution of the HVI (8); hence $x' \in \Xi$.

We further prove that $x' = x^*$, a unique solution of the THVI (7). As a matter of fact, it follows from (22) that for $x' \in \Xi$

$$\|x_{t_n} - x'\|^2 \leq \frac{1}{\mu\eta - \gamma\rho}\big\langle (\gamma f - \mu F)x', x_{t_n} - x'\big\rangle.$$

Therefore, the weak convergence to x' of $\{x_{t_n}\}$ implies that $x_{t_n} \to x'$ in norm. Now, we can let $t = t_n \to 0$ in (21) to get

$$\big\langle (\gamma f - \mu F)x', x' - y\big\rangle \geq 0, \quad \forall y \in \Xi.$$

It turns out that $x' \in \Xi$ solves THVI (7). By uniqueness, we have $x' = x^*$. This is sufficient to guarantee that $x_t \to x^*$ in norm, as $t \to 0$.

Finally, put $f = 0$ and let $\{x_{s,t}\}$ be defined by the implicit scheme (10). Then the THVI (7) reduces to VI (13). Moreover, the iterated limit in the norm topology

$$s - \lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is the unique solution $x^* \in \Xi$ of VI (13). In addition, it is easy to see that for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$. This completes the proof. □

In the above Theorem 3.1, put $\mu = 2$, $F = \frac{1}{2}I$ and $\gamma = \tau = 1$. Then the VI (8) reduces to HVI (5) and $\Xi = \Omega$. In this case, THVI (7) reduces to VI (25). In terms of Theorem 3.1, for each fixed $t \in\ ]0,1[$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in \mathrm{Fix}(T)$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution $x^* \in \Omega$ of VI (25). Hence, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Additionally, if we take $f = 0$, then VI (13) reduces to the following variational inequality:

$$\text{find } x^* \in \Omega \quad : \quad \big\langle x^*, x - x^*\big\rangle \geq 0, \quad \forall x \in \Omega,$$

which is equivalent to

$$x^* = P_\Omega(0).$$

Note that

$$x^* = P_\Omega(0) \quad \Leftrightarrow \quad \big\|0 - x^*\big\| \leq \|0 - y\| \ (\forall y \in \Omega) \quad \Leftrightarrow \quad \big\|x^*\big\| = \min_{y\in\Omega}\|y\|.$$

Thus, by Theorem 3.1, the iterated limit in the norm topology

$$s-\lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is the minimum-norm solution x^* of the HVI (5). Moreover, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$, such that the sequence $x_{s_n,t_n}\to x^*$ in norm as $n\to\infty$. Therefore, we obtain the following conclusion.

Corollary 3.1 [24, Theorem 3.1] *Let $f:C\to H$ be a ρ-contraction with coefficient $\rho\in[0,1[$ and $S,T:C\to C$ be two nonexpansive mappings with* $\mathrm{Fix}(T)\neq\emptyset$. *Suppose that the solution set Ω of the HVI (5) be nonempty. For each $(s,t)\in\ ]0,1[\times]0,1[$, let $x_{s,t}$ be defined implicitly by (11). Then, for each fixed $t\in\ ]0,1[$, the net $\{x_{s,t}\}$ converges in norm, as $s\to 0$, to a point $x_t\in\mathrm{Fix}(T)$. Moreover, as $t\to 0$, the net $\{x_t\}$ converges in norm to a unique solution x^* of the following variational inequality*:

$$\textit{find } x^*\in\Omega \quad : \quad \langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \forall x\in\Omega. \tag{25}$$

Moreover, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$, such that the sequence $x_{s_n,t_n}\to x^*$ in norm as $n\to\infty$.

In particular, if we take $f=0$ and if $x_{s,t}$ is defined by the implicit scheme (12), then the iterated limit in the norm topology

$$s-\lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is the minimum-norm solution x^* of the HVI (5). Furthermore, for each null sequence $\{s_n\}$ in $]0,1[$, there exists another null sequence $\{t_n\}$ in $]0,1[$ such that the sequence $x_{s_n,t_n}\to x^*$ in norm as $n\to\infty$.

Remark 3.1 Cianciaruso et al. [12] considered an implicit scheme which generates a net $\{x_{s,t}\}$ via the implicit way (see [12, p. 118]):

$$x_{s,t}=sf(x_{s,t})+(1-s)[tSx_{s,t}+(1-t)Tx_{s,t}], \tag{26}$$

where S and T are nonexpansive mappings from C into itself. Since in (26), the regularization is made for the nonexpansive mapping $tS+(1-t)T$ using the (self) contraction f, such a regularization depends on t and so the convergence of the scheme (26) is very complicated. But Yao et al. [24] made the regularization for the nonexpansive mapping S using the (possibly nonself) contraction f, and defined $x_{s,t}$ as the unique solution of the fixed-point equation:

$$x_{s,t}=P_C[s[tf(x_{s,t})+(1-t)Sx_{s,t}]+(1-s)Tx_{s,t}]. \tag{27}$$

Thus, the convergence of the scheme (27) is very simple. Furthermore, we make the regularization for the nonexpansive mapping S using the (possibly nonself) contraction f, and define $x_{s,t}$ as the unique solution of the fixed-point equation:

$$x_{s,t}=P_C[s\gamma(tf(x_{s,t})+(1-t)Sx_{s,t})+(I-s\mu F)Tx_{s,t}].$$

Whenever $\gamma = 1$, $\mu = 2$ and $F = \frac{1}{2}I$, our scheme reduces to the scheme (27). Beyond question, our scheme is more general and its convergence is very simple as well. Indeed, Cianciaruso et al. [12] investigated the behavior of the net $\{x_{s,t}\}$ along the curve $t = t(s)$ by distinguishing the cases of l being finite or infinite, where $l := \limsup_{s\to 0^+} t(s)/s$. In our and Yao et al. investigation for the behavior of the net $\{x_{s,t}\}$, this idea has completely been removed. The convergence analysis for the behavior of the net $\{x_{s,t}\}$ in [12] is complicated, and hence, our convergence analysis is better than the convergence analysis in [12].

4 An Explicit Scheme

In this section, we introduce an explicit scheme for finding a unique solution of the THVI (7). This scheme is indeed obtained by discretizing the implicit scheme investigated in the last section. More precisely, starting with an arbitrary initial guess $x_0 \in C$, we define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_C[\lambda_n \gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n \mu F)Tx_n], \quad \forall n \ge 0, \tag{28}$$

where the mappings S, T, F and the parameters μ, γ are the same as in Sect. 3, $\{\lambda_n\}$ and $\{\alpha_n\}$ are two sequences in $]0, 1[$. Additionally, if we take $f = 0$, then (28) reduces to the following iterative scheme:

$$x_{n+1} = P_C[\lambda_n(1-\alpha_n)\gamma Sx_n + (I - \lambda_n \mu F)Tx_n], \quad \forall n \ge 0. \tag{29}$$

In particular, whenever $\mu = 2$, $F = \frac{1}{2}I$ and $\gamma = \tau = 1$, the explicit schemes (28) and (29) reduce to the following explicit schemes, respectively,

$$x_{n+1} = P_C[\lambda_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\lambda_n)Tx_n], \quad \forall n \ge 0 \tag{30}$$

and

$$x_{n+1} = P_C[\lambda_n(1-\alpha_n)Sx_n + (1-\lambda_n)Tx_n], \quad \forall n \ge 0. \tag{31}$$

Comparing with the convergence of the implicit scheme (9), the convergence of the explicit scheme (28) seems much more subtle.

Theorem 4.1 *Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ and $\eta > 0$, respectively, $f : C \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1[$ and $S, T : C \to C$ be nonexpansive mappings with* $\mathrm{Fix}(T) \neq \emptyset$. *Let*

$$0 < \mu < 2\eta/\kappa^2 \quad \textit{and} \quad 0 < \gamma \le \tau,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that the solution set Ξ of the HVI (8) is nonempty and the following conditions hold:

(i) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \frac{\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}}{\alpha_n \lambda_n^2} = 0$ *and* $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\alpha_n \lambda_n^2 \lambda_{n-1}} = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n \lambda_n = \infty$;

(iv) *there are constants* $\bar{k} > 0$ *and* $\theta > 0$ *satisfying* $\|x - Tx\| \geq \bar{k}[d(x, \mathrm{Fix}(T))]^{\theta}$ *for each* $x \in C$;

(v) $\lim_{n\to\infty} \frac{\lambda_n^{1/\theta}}{\alpha_n} = 0$.

We have

(a) *If* $\{x_n\}$ *is the sequence generated by the scheme* (28) *and is bounded, then* $\{x_n\}$ *converges in norm to the point* $x^* \in \mathrm{Fix}(T)$ *which is a unique solution of the THVI* (7).

(b) *If* $\{x_n\}$ *is the sequence generated by the scheme* (29) *and is bounded, then* $\{x_n\}$ *converges in norm to a unique solution* x^* *of the VI of finding* $x^* \in \Xi$ *such that*

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in \Xi. \tag{32}$$

Proof We treat only case (a); that is, the sequence $\{x_n\}$ is generated by the scheme (28). Set

$$u_n = \lambda_n\gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n\mu F)Tx_n, \quad \forall n \geq 0.$$

Then we observe that

$$\begin{aligned}
&u_n - u_{n-1}\\
&\quad = \alpha_n\lambda_n\gamma[f(x_n) - f(x_{n-1})] + \lambda_n(1-\alpha_n)\gamma(Sx_n - Sx_{n-1})\\
&\qquad + [(I - \lambda_n\mu F)Tx_n - (I - \lambda_n\mu F)Tx_{n-1}]\\
&\qquad + (\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1})\gamma[f(x_{n-1}) - Sx_{n-1}]\\
&\qquad + (\lambda_n - \lambda_{n-1})(\gamma Sx_{n-1} - \mu FTx_{n-1}).
\end{aligned} \tag{33}$$

Let $M > 0$ be a constant such that

$$\sup_{n\geq 0}\{\gamma\|f(x_n) - Sx_n\| + \|\gamma Sx_n - \mu FTx_n\|\} \leq M.$$

It follows from (28) and (33) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C u_n - P_C u_{n-1}\|\\
&\leq \|u_n - u_{n-1}\|\\
&\leq \alpha_n\lambda_n\gamma\|f(x_n) - f(x_{n-1})\| + \lambda_n(1-\alpha_n)\gamma\|Sx_n - Sx_{n-1}\|\\
&\quad + \|(I - \lambda_n\mu F)Tx_n - (I - \lambda_n\mu F)Tx_{n-1}\|\\
&\quad + |\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|\gamma\|f(x_{n-1}) - Sx_{n-1}\|\\
&\quad + |\lambda_n - \lambda_{n-1}|\|\gamma Sx_{n-1} - \mu FTx_{n-1}\|\\
&\leq \alpha_n\lambda_n\gamma\rho\|x_n - x_{n-1}\| + \lambda_n(1-\alpha_n)\gamma\|x_n - x_{n-1}\| + (1 - \lambda_n\tau)\|x_n - x_{n-1}\|\\
&\quad + |\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|M + |\lambda_n - \lambda_{n-1}|M\\
&= [1 - \lambda_n(\tau - \gamma + \alpha_n\gamma(1-\rho))]\|x_n - x_{n-1}\| + M(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|
\end{aligned}$$

$$
\begin{aligned}
&+ |\lambda_n - \lambda_{n-1}|) \\
&\le [1-(1-\rho)\gamma\alpha_n\lambda_n]\|x_n - x_{n-1}\| + M(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|) \\
&= [1-(1-\rho)\gamma\alpha_n\lambda_n]\|x_n - x_{n-1}\| \\
&\quad + (1-\rho)\gamma\alpha_n\lambda_n \frac{M(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|)}{(1-\rho)\gamma\alpha_n\lambda_n}. \qquad (34)
\end{aligned}
$$

Hence, Conditions (ii) and (iii) allow us to apply Lemma 2.3 to (34) to get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

By (28), we get

$$
\begin{aligned}
\|x_{n+1} - Tx_n\| &= \|P_C u_n - P_C T x_n\| \\
&\le \|u_n - Tx_n\| \\
&= \|\lambda_n\gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n\mu F)Tx_n - Tx_n\| \\
&= \lambda_n\|\gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) - \mu F T x_n\| \\
&\to 0.
\end{aligned}
$$

Hence, $\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \to 0$. That $\omega_w(\{x_n\}) \subset \mathrm{Fix}(T)$ follows from the semiclosedness of $I - T$ by Lemma 2.2. By (34), we get

$$
\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\lambda_n} &\le [1-(1-\rho)\gamma\alpha_n\lambda_n]\frac{\|x_n - x_{n-1}\|}{\lambda_n} \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&= [1-(1-\rho)\gamma\alpha_n\lambda_n]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\
&\quad + [1-(1-\rho)\gamma\alpha_n\lambda_n]\left(\frac{\|x_n - x_{n-1}\|}{\lambda_n} - \frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}}\right) \\
&\quad + M\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\lambda_n} \\
&\le [1-(1-\rho)\gamma\alpha_n\lambda_n]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\
&\quad + \alpha_n\lambda_n\|x_n - x_{n-1}\|\frac{1}{\alpha_n\lambda_n}\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\
&\quad + M\alpha_n\lambda_n\frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|}{\alpha_n\lambda_n^2}. \qquad (35)
\end{aligned}
$$

In terms of Conditions (ii) and (iii), we can apply Lemma 2.3 to (35) to conclude

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\lambda_n} = 0. \qquad (36)$$

Rewriting (28) as

$$x_{n+1} = P_C u_n - u_n + \lambda_n \gamma(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n \mu F)Tx_n.$$

We obtain

$$\begin{aligned}x_n - x_{n+1} = u_n - P_C u_n + \alpha_n \lambda_n (\mu F - \gamma f)x_n + \lambda_n (1-\alpha_n)(\mu F - \gamma S)x_n \\ + (1-\lambda_n)(I-T)x_n + \lambda_n[(I-\mu F)x_n - (I-\mu F)Tx_n]. \quad (37)\end{aligned}$$

Set

$$y_n = \frac{x_n - x_{n+1}}{\lambda_n(1-\alpha_n)}, \quad \forall n \geq 0.$$

It can be easily seen from (37) that

$$\begin{aligned}y_n &= \frac{u_n - P_C u_n}{\lambda_n(1-\alpha_n)} + (\mu F - \gamma S)x_n + \frac{\alpha_n}{1-\alpha_n}(\mu F - \gamma f)x_n \\ &\quad + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}(I-T)x_n + \frac{1}{1-\alpha_n}[(I-\mu F)x_n - (I-\mu F)Tx_n].\end{aligned}$$

This yields that, for each $w \in \mathrm{Fix}(T)$ (noticing $x_n = P_C u_{n-1}$),

$$\begin{aligned}&\langle y_n, x_n - w\rangle \\ &\quad = \frac{1}{\lambda_n(1-\alpha_n)}\langle u_n - P_C u_n, P_C u_{n-1} - w\rangle + \langle(\mu F - \gamma S)x_n, x_n - w\rangle \\ &\qquad + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F - \gamma f)x_n, x_n - w\rangle \\ &\qquad + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle(I-T)x_n - (I-T)w, x_n - w\rangle \\ &\qquad + \frac{1}{1-\alpha_n}\langle(I-\mu F)x_n - (I-\mu F)Tx_n, x_n - w\rangle \\ &\quad = \frac{1}{\lambda_n(1-\alpha_n)}\langle u_n - P_C u_n, P_C u_n - w\rangle \\ &\qquad + \frac{1}{\lambda_n(1-\alpha_n)}\langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n\rangle \\ &\qquad + \langle(\mu F - \gamma S)w, x_n - w\rangle + \langle(\mu F - \gamma S)x_n - (\mu F - \gamma S)w, x_n - w\rangle \\ &\qquad + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle(I-T)x_n - (I-T)w, x_n - w\rangle \\ &\qquad + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F - \gamma f)x_n, x_n - w\rangle \\ &\qquad + \frac{1}{1-\alpha_n}\langle(I-\mu F)x_n - (I-\mu F)Tx_n, x_n - w\rangle. \quad (38)\end{aligned}$$

In (38), the first term is nonnegative due to Lemma 2.1, and the fourth and fifth terms are also nonnegative due to the monotonicity of $\mu F - \gamma S$ and $I - T$. We, therefore, derive from (38) that (noticing again $x_{n+1} = P_C u_n$)

$$\begin{aligned}
&\langle y_n, x_n - w\rangle \\
&\quad\geq \frac{1}{\lambda_n(1-\alpha_n)}\langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n\rangle + \langle(\mu F - \gamma S)w, x_n - w\rangle \\
&\qquad + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F - \gamma f)x_n, x_n - w\rangle \\
&\qquad + \frac{1}{1-\alpha_n}\langle(I - \mu F)x_n - (I - \mu F)Tx_n, x_n - w\rangle \\
&\quad= \langle u_n - P_C u_n, y_n\rangle + \langle(\mu F - \gamma S)w, x_n - w\rangle \\
&\qquad + \frac{\alpha_n}{1-\alpha_n}\langle(\mu F - \gamma f)x_n, x_n - w\rangle \\
&\qquad + \frac{1}{1-\alpha_n}\langle(I - \mu F)x_n - (I - \mu F)Tx_n, x_n - w\rangle.
\end{aligned} \tag{39}$$

Note that $\|x_n - Tx_n\| \to 0$ implies $\|(I - \mu F)x_n - (I - \mu F)Tx_n\| \to 0$. Also, since $y_n \to 0$ by (36), $\alpha_n \to 0$ and $\{x_n\}$ is bounded by assumption which implies that $\{u_n\}$ is bounded, we obtain from (39) that

$$\limsup_{n\to\infty}\langle(\mu F - \gamma S)w, x_n - w\rangle \leq 0, \quad \forall w \in \mathrm{Fix}(T). \tag{40}$$

This suffices to guarantee that $\omega_w(\{x_n\}) \subset \Xi$; namely, every weak limit point of $\{x_n\}$ solves the HVI (8). As a matter of fact, if $x_{n_j} \rightharpoonup \tilde{x} \in \omega_w(\{x_n\})$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}$, then we obtain from (40) that

$$\langle(\mu F - \gamma S)w, \tilde{x} - w\rangle \leq \limsup_{n\to\infty}\langle(\mu F - \gamma S)w, x_n - w\rangle \leq 0, \quad \forall w \in \mathrm{Fix}(T),$$

that is,

$$\langle(\mu F - \gamma S)w, w - \tilde{x}\rangle \geq 0, \quad \forall w \in \mathrm{Fix}(T).$$

Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\mathrm{Fix}(T) \neq \emptyset$ is closed and convex, by the Minty lemma [5] the last inequality is equivalent to the inequality (8). Thus, we get $\tilde{x} \in \Xi$.

Now, we take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\big\langle(\mu F - \gamma f)x^*, x_n - x^*\big\rangle = \lim_{j\to\infty}\big\langle(\mu F - \gamma f)x^*, x_{n_j} - x^*\big\rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \to \tilde{x}$ weakly; then $\tilde{x} \in \Xi$ as we just proved. Since x^* is a solution of the THVI (7), we get

$$\limsup_{n\to\infty}\big\langle(\mu F - \gamma f)x^*, x_n - x^*\big\rangle = \big\langle(\mu F - \gamma f)x^*, \tilde{x} - x^*\big\rangle \geq 0. \tag{41}$$

From (28), it follows that (noticing that $x_{n+1} = P_C u_n$ and $0 < \gamma \le \tau$)

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle u_n - x^*, x_{n+1} - x^*\rangle + \langle P_C u_n - u_n, P_C u_n - x^*\rangle \\
&\le \langle u_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle (I - \lambda_n \mu F)T x_n - (I - \lambda_n \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n \lambda_n \gamma \langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\gamma \langle S x_n - S x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le [1 - \lambda_n \tau + \alpha_n \lambda_n \gamma \rho + \lambda_n (1 - \alpha_n)\gamma]\|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le [1 - \alpha_n \lambda_n \gamma (1 - \rho)]\|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le [1 - \alpha_n \lambda_n \gamma (1 - \rho)]\frac{1}{2}\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + \alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

It turns out that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{1 - \alpha_n \lambda_n \gamma (1 - \rho)}{1 + \alpha_n \lambda_n \gamma (1 - \rho)}\|x_n - x^*\|^2 \\
&\quad + \frac{2}{1 + \alpha_n \lambda_n \gamma (1 - \rho)}\big[\alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle\big] \\
&\le [1 - \alpha_n \lambda_n \gamma (1 - \rho)]\|x_n - x^*\|^2 \\
&\quad + \frac{2}{1 + \alpha_n \lambda_n \gamma (1 - \rho)}\big[\alpha_n \lambda_n \langle (\gamma f - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \lambda_n (1 - \alpha_n)\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle\big]. \qquad (42)
\end{aligned}$$

However, since $x^* \in \Xi$ and by Assumption (iv), we obtain that

$$\begin{aligned}
&\langle (\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad = \langle (\gamma S - \mu F)x^*, x_{n+1} - P_{\mathrm{Fix}(T)} x_{n+1}\rangle + \langle (\gamma S - \mu F)x^*, P_{\mathrm{Fix}(T)} x_{n+1} - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \big\langle (\gamma S-\mu F)x^*, x_{n+1}-P_{\mathrm{Fix}(T)}x_{n+1}\big\rangle \\
&\le \big\|(\gamma S-\mu F)x^*\big\| d(x_{n+1},\mathrm{Fix}(T)) \\
&\le \big\|(\gamma S-\mu F)x^*\big\| \left(\frac{1}{\bar{k}}\|x_{n+1}-Tx_{n+1}\|\right)^{1/\theta}. \qquad (43)
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\|x_{n+1}-Tx_{n+1}\| &\le \|x_{n+1}-Tx_n\| + \|Tx_n-Tx_{n+1}\| \\
&\le \|x_n-x_{n+1}\| + \lambda_n\|\gamma(\alpha_n f(x_n)+(1-\alpha_n)Sx_n)-\mu FTx_n\| \\
&= \|x_n-x_{n+1}\| + \lambda_n\|\gamma\alpha_n(f(x_n)-Sx_n)+\gamma Sx_n-\mu FTx_n\| \\
&\le \|x_n-x_{n+1}\| + M\lambda_n. \qquad (44)
\end{aligned}$$

Hence, for a big enough constant $\bar{k}_1>0$, we have

$$\begin{aligned}
\big\langle (\gamma S-\mu F)x^*, x_{n+1}-x^*\big\rangle &\le \bar{k}_1(\lambda_n+\|x_n-x_{n+1}\|)^{1/\theta} \\
&\le \bar{k}_1\lambda_n^{1/\theta}\left(1+\frac{\|x_n-x_{n+1}\|}{\lambda_n}\right)^{1/\theta}. \qquad (45)
\end{aligned}$$

Combining (42)–(45), we get

$$\begin{aligned}
\big\|x_{n+1}-x^*\big\|^2 &\le [1-\alpha_n\lambda_n\gamma(1-\rho)]\big\|x_n-x^*\big\|^2 \\
&\quad + \frac{2}{1+\alpha_n\lambda_n\gamma(1-\rho)}\Big[\alpha_n\lambda_n\big\langle(\gamma f-\mu F)x^*, x_{n+1}-x^*\big\rangle \\
&\quad + \lambda_n(1-\alpha_n)\big\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\big\rangle\Big] \\
&\le [1-\alpha_n\lambda_n\gamma(1-\rho)]\big\|x_n-x^*\big\|^2 \\
&\quad + \frac{2\alpha_n\lambda_n}{1+\alpha_n\lambda_n\gamma(1-\rho)}\Bigg[\big\langle(\gamma f-\mu F)x^*, x_{n+1}-x^*\big\rangle \\
&\quad + \frac{\bar{k}_1\lambda_n^{1/\theta}}{\alpha_n}\left(1+\frac{\|x_n-x_{n+1}\|}{\lambda_n}\right)^{1/\theta}\Bigg] \\
&= (1-\gamma_n)\big\|x_n-x^*\big\|^2+\delta_n, \qquad (46)
\end{aligned}$$

where

$$\gamma_n = \alpha_n\lambda_n\gamma(1-\rho)$$

and

$$\begin{aligned}
\delta_n &= \frac{2\alpha_n\lambda_n}{1+\alpha_n\lambda_n\gamma(1-\rho)} \\
&\quad \times\left[\big\langle(\gamma f-\mu F)x^*, x_{n+1}-x^*\big\rangle + \frac{\bar{k}_1\lambda_n^{1/\theta}}{\alpha_n}\left(1+\frac{\|x_n-x_{n+1}\|}{\lambda_n}\right)^{1/\theta}\right].
\end{aligned}$$

Now Condition (iii) implies that $\sum_{n=0}^{\infty} \gamma_n = \infty$, and (v) and (41) imply that

$$\limsup_{n\to\infty} \delta_n/\gamma_n \le 0.$$

Therefore, we can apply Lemma 2.3 to (46) to conclude that $x_n \to x^*$. The proof of part (a) is complete. It is easy to see that part (b) now becomes a straightforward consequence of part (a) since, if $f = 0$, THVI (7) reduces to VI (32). This completes the proof. □

In the above Theorem 4.1, put $\mu = 2$, $F = \frac{1}{2}I$ and $\gamma = \tau = 1$. Then the HVI (8) reduces to the HVI (5) and $\Xi = \Omega$. In this case, the THVI (7) reduces to VI (47). In terms of Theorem 4.1 (a), $\{x_n\}$ converges in norm to the point $x^* \in \mathrm{Fix}(T)$ which is a unique solution of VI (47).

Additionally, if we take $f = 0$, then VI (32) reduces to the following variational inequality:

$$\text{find } x^* \in \Omega \quad : \quad \langle x^*, x - x^* \rangle \ge 0, \quad \forall x \in \Omega,$$

which is equivalent to

$$x^* = P_\Omega(0).$$

Note that

$$x^* = P_\Omega(0) \quad \Leftrightarrow \quad \|0 - x^*\| \le \|0 - y\| \ (\forall y \in \Omega) \quad \Leftrightarrow \quad \|x^*\| = \min_{y\in\Omega} \|y\|.$$

Thus, by Theorem 4.1 (b), $\{x_n\}$ converges in norm to the minimum-norm solution of the HVI (5). Therefore, we get the following conclusion.

Corollary 4.1 [24, Theorem 4.1] *Let $f : C \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1[$ and $S, T : C \to C$ is two nonexpansive mappings with* $\mathrm{Fix}(T) \ne \emptyset$. *Assume that the solution set Ω of the HVI* (5) *be nonempty and that the following conditions hold*:

(i) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \frac{\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}}{\alpha_n\lambda_n^2} = 0$ *and* $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\alpha_n\lambda_n^2\lambda_{n-1}} = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n\lambda_n = \infty$;
(iv) *there are constants $\bar{k} > 0$ and $\theta > 0$ satisfying $\|x - Tx\| \ge \bar{k}[d(x, \mathrm{Fix}(T))]^\theta$ for each $x \in C$*;
(v) $\lim_{n\to\infty} \frac{\lambda_n^{1/\theta}}{\alpha_n} = 0$.

We have

(a) *If $\{x_n\}$ is the sequence generated by the scheme* (30) *and is bounded, then $\{x_n\}$ converges in norm to the point $x^* \in \mathrm{Fix}(T)$ which is a unique solution of the variational inequality*:

$$\textit{find } x^* \in \Omega \quad : \quad \langle (I - f)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \Omega. \tag{47}$$

(b) *If* $\{x_n\}$ *is the sequence generated by the scheme* (31) *and is bounded*, *then* $\{x_n\}$ *converges in norm to a minimum-norm solution of the HVI* (5).

Remark 4.1 As pointed out in [24], whenever the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen as

$$\alpha_n = \frac{1}{(n+1)^\alpha} \quad \text{and} \quad \lambda_n = \frac{1}{(n+1)^\lambda},$$

Conditions (i)–(iii) of Theorem 4.1 are satisfied provided $0 < \alpha, \lambda < 1$ and $\alpha + 2\lambda \leq 1$. Also, Condition (v) is satisfied provided $\lambda/\alpha > \theta$.

5 Concluding Remarks

We considered a variational inequality with variational inequality constraint over a set of fixed points of a nonexpansive mapping, called triple hierarchical variational inequality (THVI). An example of THVI is also given. We combined the regularization method, the hybrid steepest-descent method, and the projection method to propose an implicit scheme that generates a net in an implicit way, and studied its strong convergence to a unique solution of THVI. We also proposed an explicit scheme that generates a sequence via an iterative algorithm and proved that this sequence converges strongly to a unique solution of THVI.

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References

1. Luo, Z.Q., Pang, J.S., Ralph, D.: Mathematical Programs with Equilibrium Constraints. Cambridge University Press, New York (1996)
2. Outrata, J., Kočvara, M., Zowe, J.: Nonsmooth Approach to Optimization Problems with Equilibrium Constraints. Kluwer Academic, Dordrecht (1998)
3. Glowinski, R., Lions, J.L., Tremoliers, R.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
4. Jaillet, P., Lamberton, D., Lapeyre, B.: Variational inequalities and the pricing of American options. Acta Appl. Math. **21**, 263–289 (1990)
5. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
6. Oden, J.T.: Qualitative Methods on Nonlinear Mechanics. Prentice-Hall, Englewood Cliffs (1986)
7. Zeidler, E.: Nonlinear Functional Analysis ant Its Applications, III: Variational Methods and Applications. Springer, New York (1985)
8. Combettes, P.L.: A block iterative surrogate constraint splitting method for quadratic signal recovery. IEEE Trans. Signal Process. **51**(7), 1771–1782 (2003)
9. Slavakis, K., Yamada, I.: Robust wideband beamforming by the hybrid steepest descent method. IEEE Trans. Signal Process. **55**, 4511–4522 (2007)
10. Iiduka, H.: Fixed point optimization algorithm and its application to power control in CDMA data networks. Math. Program. (2011). doi:10.1007/s10107-010-0427-x

11. Cabot, A.: Proximal point algorithm controlled by a slowly vanishing term: Applications to hierarchical minimization. SIAM J. Optim. **15**, 555–572 (2005)
12. Cianciaruso, F., Colao, V., Muglia, L., Xu, H.K.: On an implicit hierarchical fixed point approach to variational inequalities. Bull. Aust. Math. Soc. **80**(1), 117–124 (2009)
13. Iiduka, H.: A new iterative algorithm for the variational inequality problem over the fixed point set of a firmly nonexapnsive mapping. Optimization **59**, 873–885 (2010)
14. Iiduka, H., Yamada, I.: A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping. SIAM J. Optim. **19**, 1881–1893 (2009)
15. Mainge, P.-E., Moudafi, A.: Strong convergence of an iterative method for hierarchical fixed-points problems. Pac. J. Optim. **3**(3), 529–538 (2007)
16. Marino, G., Xu, H.K.: Explicit hierarchical fixed point approach to variational inequalities. J. Optim. Theory Appl. **149**(1), 61–78 (2011)
17. Moudafi, A.: Krasnoselski-Mann iteration for hierarchical fixed-point problems. Inverse Probl. **23**, 1635–1640 (2007)
18. Moudafi, A., Mainge, P.-E.: Towards viscosity approximations of hierarchical fixed-point problems. Fixed Point Theory Appl. (2006) 1–10. Article ID 95453 (2006)
19. Xu, H.K.: Viscosity methods for hierarchical fixed point approach to variational inequalities. Taiwan. J. Math. **14**(2), 463–478 (2010)
20. Yamada, I.: The hybrid steepest descent method for the variational inequality problems over the intersection of fixed point sets of nonexpansive mappings. In: Butnariu, D., Censor, Y., Reich, S. (eds.) Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications. Stud. Comput. Math., vol. 8, pp. 473–504. North-Holland, Amsterdam (2001)
21. Yamada, I., Ogura, N.: Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings. Numer. Funct. Anal. Optim. **25**, 619–655 (2004)
22. Yamada, I., Ogura, N., Shirakawa, N.: A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems. In: Inverse Problems, Image Analysis, and Medical Imaging. Contemp. Math., vol. 313, pp. 269–305. Amer. Math. Soc., Providence (2002)
23. Yao, Y., Liou, Y.-C.: Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed point problems. Inverse Probl. **24**(1), 015015 (2008)
24. Yao, Y., Chen, R., Xu, H.K.: Schemes for finding minimum-norm solutions of variational inequalities. Nonlinear Anal. **72**, 3447–3456 (2010)
25. Iidika, H.: Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem. Nonlinear Anal. **71**, 1292–1297 (2010)
26. Iidika, H.: Iterative algorithm for solving triple-hierarchical constrained optimization problem. J. Optim. Theory Appl. **148**, 580–592 (2011)
27. Lu, X.W., Xu, H.K., Yin, X.M.: Hybrid methods for a class of monotone variational inequalities. Nonlinear Anal. **71**, 1032–1041 (2009)
28. Goebel, K., Kirk, W.A.: Topics on Metric Fixed-Point Theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
29. Xu, H.K., Kim, T.H.: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**(1), 185–201 (2003)
30. Xu, H.K.: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279–291 (2004)